A Criterion that a Function Belong to All Free Maximal Ideals of $C(X)$

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Let $C(X)$ denote the ring of real valued continuous functions defined on a space X. The zero set, $Z(f)$ of a function f in $C(X)$ is the set of points x in X for which $f(x)=0$. An ideal of $C(X)$ is called a free ideal if the intersection of zero sets of functions in the ideal is empty.

The main result of this note states that if X is a Tychonoff space then a function f in $C(X)$ belongs to all free maximal ideals of $C(X)$ if and only if the cozero set, $Coz(f)=X-Z(f)$, is locally compact, σ-compact, and totally bounded relative to any uniform structure admissible to X. Using this result, we characterize the spaces X in which the intersection of free maximal ideals of $C(X)$ agrees with the subring $C_K(X)$ of $C(X)$ consisting of functions with compact supports. It is also proven that if X is a totally ordered space then $C_K(X)$ coincides with the intersection of free maximal ideals of $C(X)$.

We begin by noting the following obvious

Lemma 1. *Let X be a Tychonoff space and let Y be a subset of X. If there is continuous pseudometric d on X such that the d-distance between any two distinct points in Y is greater than a preassigned positive number, then Y is C-embedded in X, i.e., any function in $C(Y)$ extends to a function in $C(X)$.*

Recall the Gelfand-Kolmogoroff theorem which asserts that there is a one-to-one map $p\to M_p$ from the Stone-Čech compactification βX of X onto the set of maximal ideals of $C(X)$, where M_p consists of those f in $C(X)$ for which p belongs to the closure in βX of $Z(f)$. The maximal ideal M_p is free if and only if p fails to be in X.

Lemma 2. *Let X be a Tychonoff space. If f belongs to every free maximal ideal of $C(X)$, then the support of f is totally bounded with respect to any admissible uniform structure for X.*

Proof. Suppose contrary to the statement of this lemma that there is a uniform structure $\mathscr{D}$ admissible to X such that the support, $K=\mathrm{cl}_X\, Coz(f)$, of f fails to be totally bounded with respect to $\mathscr{D}$. Then there exist an infinite subset Y of K, a pseudometric d in $\mathscr{D}$ and a positive number e satisfying the condition that the d-distance between any two distinct points in Y is greater than e. For each y in K pick a point y' in $Coz(f)$ with $d(y, y')<e/3$, and let Y' denote the set of these y'. Since Y' is C-embedded in X by Lemma 1, it is completely separated from $Z(f)$ by Theorem

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1.18 of [1]. Thus, the noncompact set Y' has a limit point p in $\beta X-X$ which is not in $\mathrm{cl}_{\beta X} Z(f)$, and f cannot be in the free maximal ideal M_p. This contradiction proves the lemma.

Let $C_K(X)$ denote the ring of functions in $C(X)$ with compact supports. As an immediate consequence of Lemma 1, we obtain the following result of Robinson [3].

COROLLARY 1. *If the space X admits a complete uniform structure, then the intersection of free maximal ideals in* $C(X)$ *is identical with the ring* $C_K(X)$.

Proof. A totally bounded closed subset of a complete uniform space is necessarilly compact.

REMARK. In anglogy with this corollary, one might be tempted to assert that if X admit a sequentially complete uniform structure then a function in the intersection of free maximal ideals of $C(X)$ would have a sequentially or countably compact support. Unfortunately, this cannot be the case even when X is locally compact as we shall see in the following argument.

Let N denote the space of natural numbers, let W be the space of ordinals less than the first uncountable ordinal Ω and let $W^* = W \cup \{\Omega\}$, all with order topology. Define X to be the subspace $(\beta N\times W)\cup(N\times\Omega)$ of $\beta N\times W^*$. Clearly, X is locally compact. If Y is a discrete closed subset of X, then countable compactness of $\beta N\times W$ implies that Y intersects $\beta N\times W$ in at most a finite number of points. Since $Y\cap(N\times\Omega)$ must be C^*-embedded in $\beta N\times\Omega$, this means that Y is C^*-embedded in $Y\cup(\beta N\times\Omega)$, which is closed in $\beta N\times W^*$. Since $Y\cup(\beta N\times\Omega)$ is C^*-embedded in $\beta N\times W^*$ by Tietze extension theorem, it follows that Y is C^*-embedded in $\beta N\times W^*$. In turn, this implies that Y is C^*-embedded in X, and X admits a sequentially complete uniform structure by [2].

Now, let $f\in C^*(N\times W^*)$ be defined by $f(n, \alpha)=1/n$, $\alpha\in W^*$. Evidently, f extends to a function in $C^*(X)$ which we shall also denote by f. By the Gelfand-Kolmogoroff theorem, f belongs to the intersection of all free maximal ideals of $C(X)$. However, the support of f is the whole space X, which is not countably compact since it has the infinite discrete closed subset $N\times\Omega$.

In spite of the above remark, one may readily deduce from Lemma 1 and the Tietze theorem that if X is a normal Hausdorff space then every function in the intersection of free maximal ideals of $C(X)$ has a countably compact support. In fact, we can state something more than this. Namely,

COROLLARY 2. *If X is a Tychonoff space in which countable discrete closed sets are C-embedded then every function in the intersection of free maximal ideals of* $C(X)$ *has a countably compact support.*

The proof of this result is omitted as it is immediate from Lemma 2. It is to be noted here that countably paracompact spaces satisfy the condition of Corollary 2.

LEMMA 3. *Let X be a Tychonoff space. If f is a function belonging to all free maximal ideals of $C(X)$ then $Coz(f)$ is locally compact and σ-compact.*

Proof. For each poistive integer n, let F_n be the set of points x in X with $|f(x)| \geq 1/n$. Each F_n is completely separated from $Z(f)$, and so the sets F_n and $Z(f)$ have disjoint closures in βX. Since $\beta X - X$ must be contained in $\mathrm{cl}_{\beta X} Z(f)$ by the Gelfand-Kolmogoroff theorem, $\mathrm{cl}_{\beta X} F_n$ is contained in X. This implies, however, that F_n is compact because F_n is closed in X and it must be closed in βX as well. Lemma 3 then follows from the facts: (i) F_{n+1} is a neighborhood of F_n for each n, and (ii) $Coz(f)$ is the sum of the sets F_n.

We now state the main result of this paper:

THEOREM 1. *Let X be a Tychonoff space. In order that a function in $C(X)$ be in the intersection of free maximal ideals of $C(X)$, it is necessary and sufficient that the following two conditions be satisfied:*

(1) $Coz(f)$ is totally bounded relative to any uniform structure admissible to X, and

(2) $Coz(f)$ is locally compact and σ-compact.

Proof. The necessity part follows from Lemmas 2 and 3. To prove the sufficiency, let f be a function in $C(X)$ with property (2) which fails to be in some free maximal ideal M_p, $p \in \beta X - X$. Since p does not belong to $\mathrm{cl}_{\beta X} Z(f)$, there are open subsets U, V of βX such that $p \in U$, $\mathrm{cl}_{\beta X} U \subset V$, and $\mathrm{cl}_{\beta X} V \cap \mathrm{cl}_{\beta X} Z(f) = \phi$. Let F and W denote, respectively, the intersections of X with the sets $\mathrm{cl}_{\beta X} U$ and V. Then $p \in \mathrm{cl}_{\beta X} F$, $F \subset W$, and W is completely separated from $Z(f)$. For each $x \in F$, let G_x be an open neighborhood of x whose closure is compact and contained in W, and let $\mathscr{G}$ be the open cover of $Coz(f)$ consisting of $Coz(f) - F$ and the sets G_x, $x \in F$. Since $Coz(f)$ is paracompact by (2), there is an open locally finite refinement $\mathscr{H}$ of $\mathscr{G}$. Moreover, we may assume without loss of generality that $\mathscr{H}$ is a countable cover as $\mathrm{Coz}(f)$ must be Lindelöf by (2). Let $\mathscr{H}'$ denote the collection of those members of $\mathscr{H}$ which meet F. Clearly, $\mathscr{H}'$ is locally finite on $Coz(f)$. It is locally finite on X too because each member of $\mathscr{H}'$ is contained in W and $X - W$ is a neighborhood of $Z(f)$ disjoint from all members of $\mathscr{H}'$. It should also be noted here that $\mathscr{H}'$ cannot be a finite collection since $p \in \mathrm{cl}_{\beta X} F - F$ implies F is noncompact but $\mathscr{H}'$ is a locally finite cover of F consisting of sets with compact closures. Hence, we may suppose that members of $\mathscr{H}'$ are indexed by the set of natural numbers, that is, we may write $\mathscr{H} = \{H_n\}$, $n = 1, 2, \cdots\cdots$.

From each H_n, pick a point x_n and let g_n be a nonnegative function in $C(X)$ with $g_n(x_n) = n$ and $g_n(x) = 0$ for x in $X - H_n$, and define g by letting $g(x)$ to be the sum of $g_n(x)$. Local finiteness of $\mathscr{H}'$ implies that g is a well defined function in $C(X)$. However, since g is unbounded on $Coz(f)$, the statement (2) cannot be true. This completes the proof.

REMARK. The proof of Theorem 1 given above can be shortened in some degree by showing that F is C-embedded in X. When this is done, since F is realcompact but not compact, existence of unbounded functions in $C(F)$ guarantees that F fails to be totally bounded.

The following theorem characterizes the spaces in which the intersection of free maximal ideals of $C(X)$ and the ring $C_K(X)$ of functions with compact supports coincide.

THEOREM 2. *Let X be a Tychonoff space. In order that the intersection of free maximal ideals of $C(X)$ be identical with the ring $C_K(X)$, it is necessary and sufficient that the following be valid:*

If a locally compact and σ-compact open subset of X is totally bounded with respect every uniform structure admissible to X then it has a compact closure.

Proof. In view of Theorem 1, all we have to do is that if U is a subset of X subject to the conditions stated in the theorem then $U = Coz(f)$ for some f in $C(X)$. To do this, let F_1, F_2, ... be compact sets with $\bigcup F_n = U$. since U is locally compact and open in X there is an open set U_1 containing F_1 such that the closure of U_1 is compact and contained in U. Suppose that we have constructed open sets $U_1, \cdots\cdots, U_n$ with compact closures contained in U such that (i) U_k contains $F_1 \cup \cdots\cdots \cup F_k$ for $k=1, \cdots\cdots, n$, and (ii) U_{k+1} contains the closure of U_k for $k=1, \cdots\cdots, n-1$. We then let U_{n+1} be an open set containing the compact set $F_{n+1} \cup \mathrm{cl}\, U_n$ such that the closure of U_{n+1} is compact and contained in U. The sets $U_1, \cdots\cdots, U_{n+1}$ also satisfy the conditions (i) and (ii), and we obtain a sequence of open sets such that $\mathrm{cl}\, U_n \subset U_{n+1}$ and $\bigcup U_n = U$. For each positive integer n, let f_n be a continous function from U into the interval $[0, 2^{-n}]$ such that $f_n(x) = 2^{-n}$ for $x \in \mathrm{cl}\, U_n$ and $f_n(x) = 0$ for $x \in U - U_{n+1}$. Evidently, each f_n extends to a function in $C(X)$ by letting its value be zero on $X-U$. This we assume done and denote the extended function still by f_n.

Now, for each x in X, define $f(x)$ to be the sum of the infinite series $f_1(x) + f_2(x) + \cdots\cdots$. Since this serires conveges to $f(x)$ uniformly on X, f is in $C(X)$. For this f, we have $U = Coz(f)$ as desired.

As another application of Theorem 1, we have the following.

THEOREM 3. *If X is a totally ordered space, then the intersection of free maximal ideals of $C(X)$ coincides with $C_K(X)$.*

Proof. Let f be a function in the intersection of free maximal ideals of $C(X)$, let K denote the support of f, and let Y be a totally ordered compactification of X. It is enough to show that the closure in Y of $Coz(f)$ is contained in K.

By Theorem 1, there are compact sets F_1, F_2, ..., such that the sum of sets F_n is $Coz(f)$. As in the proof of Theorem 2, we may assume that each F_n is contained in the interior of F_{n+1}. If y is a point in $Y-K$, then let K' be the set of points in

K which precede y. Then the set K' is both open and closed in K. If K' is not empty, it is obvious that K' intersects with $Coz(f)$ in a nonempty set. Let k be the smallest integer such that F_k meets K', and let x_n be the least upper bound of $K' \cap F_{n+k-1}$. By compactness, x_n belongs to $K' \cap F_{n+k-1}$, and the set consisting of these x_n is a discrete set which is closed and cofinal in $K' \cap Coz(f)$. This set has a limit point x in K' since K' is closed in K but, because of normality of X, K is countably compact by Corollary 2 to Lemma 2. This x is easily seen to be the least upper bound of K', and set of points in Y preceded by x is a neighborhood of y disjoint from K'. Accordingly, y is not a limit point of K'. Since, similar argument shows that y is not a limit point of $K-K'$, y cannot be a limit point of K. This shows that K is closed in Y, and the proof is complete.

References

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