A New Proof of a Theorem in the Matrix Ring

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1. INTRODUCTION. Let R be a ring and R_n be the ring of $n \times n$ matrices over R. If we denote the Jacobson radical of R by $J(R)$, then $J(R)_n = J(R_n)$ which is a well known fact. [1, p. 11] Its proof makes very strong use of the notion of quasi-regularity. In the weak radical theory [2] K. Koh and A.C. Newborn stated that they had no analogue of this notion and proofs in the weak radical theory suggest proofs of the classical theorem which do not depend on quasi-regularity. The purpose of this note is to prove $J(R)_n = J(R_n)$ employing the notion of a modular one-sided ideal which corresponds to that of an almost maximal one-sided ideal.

2. THEOREM. $J(R_n) = J(R)_n$.

Proof. We shall prove $J(R_n) \subseteq J(R)_n$. Let $\alpha = (a_{ij}) \in R_n$, $\alpha \notin J(R)_n$. Assume $a_{kh} \notin J(R)$. Then there exists a modular maximal right ideal I of R such that $a_{kh} \notin I$. Let $I^* = \{(b_{ij}) \notin R_n | b_{kj} \in I,\ 1 \leq j \leq n\}$. We show that I^* is a modular maximal right ideal of R_n. Let $(r_{ij}) \notin I$. We shall prove that $I^* + (r_{ij})R_n = R_n$, which proves that I^* is a maximal right ideal of R_n. Clearly $r_{kh} \notin I$ for some h. Consider $I + r_{kh}R$. If $I + r_{kh}R \neq R$, then $I + r_{kh}R = I$ because I is maximal. Therefore $r_{kh}R \subseteq I$. Let $J = \{x \in R | xR \subseteq I\}$. Clearly J is a right ideal of R containing I. Since $r_{kh} \in J$, $r_{kh} \notin I$, $J = R$. Hence $R^2 \subseteq I$. Let e be a left identity modulo I. Then $a \in I$ for all $a \in R$. This is impossible. Therefore $I + r_{kh}R = R$. Let $\beta = (y_{ij})$ be any matrix of R_n. Then there exist s_{k1}, $s_{k2}, \cdots, s_{kn}$ in I and t_{h1}, $t_{h2}, \cdots, t_{hn}$ in R such that $s_{ki} + r_{kh}t_{hi} = y_{ki}$ for $1 \leq i \leq n$. Let γ denote the matrix having its k-th row as $(s_{k1}, s_{k2}, \cdots\cdots, s_{kn})$ and the remaining elements zero. Then $\gamma \in I^*$. Let δ be the matrix whose h-th row is $(t_{h1}, t_{h2}, \cdots\cdots, t_{hn})$ and the remaining elements zero. Then the k-th row of $\gamma + (r_{ij})\delta$ is $(y_{k1}, y_{k2}, \cdots\cdots, y_{kn})$. Therefore $(y_{ij}) - [\gamma + (r_{ij})\delta] \in I^*$. Hence $(y_{ij}) \in I^* + (r_{ij})R_n$, since $\gamma + (r_{ij})\delta \in I^* + (r_{ij})R$. It follows that $I^* + (r_{ij})R_n = R_n$. Moreover, eE_{kk} is a left identity modulo I^* since for every element σ of R_n, $\sigma - eE_{kk}\sigma \in I^*$. Since $\alpha \notin I^*$, it follows that $\alpha \notin J(R_n)$.

To prove that $J(R)_n \subseteq J(R_n)$, suppose $\alpha = (a_{ij}) \in J(R)_n$. We will show that $\alpha \in J(R_n)$. We can suppose α to be a matrix whose every element is zero except the k-th row element, $1 \leq k \leq n$. Suppose $\alpha \notin J(R_n)$. Then there exists a modular maximal right ideal J^* such that $\alpha \notin J^*$. Let ε be a left identity modulo J^*. Then there exists an element β' in R_n such that $(\alpha + J^*)\beta' = \varepsilon + J^*$ since $M = R_n - J^*$ is an irreducible R_n-module. Set $\beta = \alpha\beta'$. Then $\beta - \varepsilon \in J^*$. It follows that β is a left identity modulo J^*.

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It is easily seen that $\beta \in N(J^*)$, $\beta \notin J^*$. We will show that $\beta \notin J(R)_n$ and this fact proves that $\alpha \notin J(R)_n$. Let $\gamma = b_{kk}E_{kk}$. Then $\gamma\beta = \beta^2$. But $\beta^2 \notin J^*$ since $N(J^*)-J^*$ is a division ring. Hence $\gamma\beta \notin J^*$ and consequently $\gamma \notin J^*$. Let $I^* = \{\rho \in R_n \mid \gamma\rho \in J^*\}$. Then I^* is a modular maximal right ideal in R_n by the Proposition 2 in [1, p. 6]. If $\rho \in I^*$, $\gamma\rho \in J^*$. Since $\beta \in N(J^*)$, $\beta\gamma\rho \in J^*$. But $\beta\gamma\rho = \gamma^2\rho$ since $\beta\gamma = \gamma^2$. Hence $\gamma\rho \in I^*$. Therefore $\gamma \in N(I^*)$. Now $\gamma \notin I^*$. For, suppose $\gamma \in I^*$, then $\beta\gamma \in J^*$ and this implies that $\gamma \in J^*$, which is a contradiction. Let $I = \{r \in R \mid b_{kk}rE_{kk} \in J^*\}$. Then I is a right ideal in R. Now $b_{kk} \notin I$; for suppose $\rho = (r_{ij}) \in I^*$, and $r_{kk} = b_{kk}$. We can assume $r_{ij} = 0$ if $i \neq k$. Then $\gamma^2 = \rho\gamma$, which is impossible because $\gamma \in N(I^*)$, $\gamma \notin I^*$. If we show that I is modular maximal, then $\beta \notin J(R)_n$.

We show now that I is modular maximal. Let $x \notin I$. Then $b_{kk}xE_{kk} \notin J^*$. It follows that $J^* + b_{kk}xE_{kk}R_n = R_n$ due to the fact that J^* is maximal and $\gamma \notin I^*$. Therefore there exist (z_{ij}) in J^* and (t_{ij}) in R_n such that $b_{kk}E_{kk} = (z_{ij}) + b_{kk}xE_{kk}(t_{ij})$. Let $y \in R$. Post-multiplying by yE_{kk}, we get $b_{kk}yE_{kk} = b_{kk}xt_{kk}yE_{kk} + (z_{ij})yE_{kk}$, so that $b_{kk}(y - xt_{kk}y)E_{kk}$ is in J^*. It follows that $y - xt_{kk}y \in I$, $y \in I + xR$, and $I + xR = R$. Thus I is maximal. We show now that b_{kk} is a left identity modulo I. Let $a \in R$. Then $aE_{kk} - \beta aE_{kk} \in J^*$ since β is a left identity modulo J^*. It follows that $\beta aE_{kk} - \beta^2 aE_{kk} \in J^*$ because $\beta \in N(J^*)$. But $\beta aE_{kk} - \beta^2 aE_{kk} = b_{kk}(a - b_{kk}a)E_{kk}$. Hence $a - b_{kk}a \in I$. Q.E.D.

References

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