Mappings Generating Upper-semicontinuous Decompositions of Spaces with Coherent Topologies

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1. Introduction

In this paper we generalize the concepts of compactly generated spaces (or Hausdorff k-spaces) and reflexive compact mappings. Using these concepts we obtain sufficient conditions for mappings to generate upper-semicontinuous (u.s.c.) decompositions of certain types of spaces with coherent topologies. We also show that some of the results generalize similar situations which are given previously.

2. Terminologies

A *k-space* X is a topological space having a topology coherent with the collection of its closed compact subsets; i.e., if a subset A of X intersects each closed compact set in a closed set, then A is closed [4]. A Hausdorff k-space is said to be ***compactly generated*** [6]. A topological space X is said to be ***locally paracompact*** if every point of X has a closed neighborhood which is a paracompact subspace of X [2]. Throughout this paper X and Y will represent topological spaces and f will be a mapping (continuous function) of X into Y. A mapping f of X into Y is said to be ***reflexive compact*** provided that $f^{-1}f(C)$ is compact for every compact subset C of X [3]. All terminologies given above will be generalized in Sections 3 and 4. A mapping f of X into Y is said to ***generate an u.s.c. decomposition*** of X if for every open set U of X, the union V of point inverses $f^{-1}(y)$ contained in U is an open subset of X.

3. P-generated spaces

Throughout this paper, a topological property P is said to be *admissible* if it is inherited by closed sets.

3.1. DEFINITION. Let X be a space and P an admissible property. A *P-set* in X is a closed subset of X which possesses the property P. X is said to be ***P-generated*** if it has a topology coherent with the collection of its P-sets; i.e., if a subset A of X intersects each P-set in a closed set, then A is closed.

3.2. DEFINITION. Let P and Q be admissible properties. We shall write $P \subset Q$ if and only if every P-set is a Q-set.

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For instance, let P and Q be the compactness and the paracompactness, respectively. Then $P \subset Q$ in a Hausdorff space.

3.3. Definition. A neighborhood which is a P-set will be called *P-neighborhood.* A space X is said to be *locally P* provided that every point of X has a P-neighborhood.

It is obvious that if $P \subset Q$ and X is locally P, then X is locally Q.

3.4. Proposition. *A locally P-space is P-generated.*

Proof. Let B be a non-closed subset of X and suppose x is an accumulation point of B which does not belong to B. Since X is locally P there is a P-neighborhood U of x and the intersection $B \cap U$ is not closed because x is an accumulation point but not a member of $B \cap V$.

3.5. Corollary. *A locally compact Hausdorff space is compactly generated* [4].

3.6. Proposition. *If a space X is P-generated and $P \subset Q$, then X is Q-generated.*

Proof. Let B be a non-closed subset of X. If X is P-generated there is a P-set V of X such that $V \cap B$ is not closed. Since $P \subset Q$, V is also Q-set.

We can define an analogue to compactly generated spaces.

3.7. Definition. *A paracompactly generated space* is a Hausdorff space having a topology coherent with the collection of its closed paracompact subsets.

From Propositions 3.4 and 3.6 we obtain

3.8. Corollary. *A Hausdorff space which is either locally paracompact or compactly generated is paracompactly generated.*

It is established that a Hausdorff space which satisfies one of the following conditions is compactly generated.

(a) locally compact [4]

(b) the first axiom of countability [4]

(c) product of a compactly generated space and a locally compact Hausdorff space [1, 6]

Hence, from the latter part of Corollary 3.8, we obtain

3.9. Corollary. *A Hausdorff space which satisfies one of the conditions (a), (b) and (c) is paracompactly generated.*

In summary we have

$$\begin{array}{ccccc}
\text{compact} & \Longrightarrow & \text{locally compact} & \Longrightarrow & \text{compactly generated} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{paracompact} & \Longrightarrow & \text{locally paracompact} & \Longrightarrow & \text{paracompactly generated} \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{normal} & \Longrightarrow & \text{completely regular} & \Longrightarrow & \text{Hausdorff}
\end{array}$$

in the class of Hausdorff spaces and we know none of the arrows can be reversed.

Especially, the following example in [1] which shows the product of two compactly generated spaces need not be a compactly generated space serves as a paracompactly generated space which is not compactly generated.

3.10. Example. Let X be the dual space of an infinite dimensional Fréchet space with the compact-open topology is compactly generated space which is not locally compact. $F=C(X, [0, 1])$ with the compact-open topology is metrizable. However $X\times F$ is not compactly generated. It follows from [5, Prop. 4] that $X\times F$ is paracompact which implies it is paracompactly generated.

4. Reflexive P-mappings

4.1. Definition. Let f be a mapping of X into Y. f is called a *reflexive P-mapping* provided that $f^{-1}f(P)$ is a P-set for every P-set P of X.

With the above definition we obtain a sufficient condition for mappings to generate u.s.c. decompositions of P-generated spaces.

4.2. Theorem. *Let X be a P-generated space and f a mapping of X into Y. If f is a reflexive P-mapping then f generates an upper-semicontinuous decomposition of X.*

This is a generalization of [3, Theorem 1] and the following proof is a slight modification of that in [3].

Proof. Let U be an open set in X containing a point inverse and V the union of the point inverses contained in U. We show that $\bar{u}-V$ is closed in X which in turn implies V is open in X and consequently f generates an u.s.c. decomposition. Let P be a P-set in X such that $H=(\bar{U}-V)\cap P\neq\phi$. For each point x in H, $f^{-1}f(x)\cap(X-U)\neq\phi$. Thus we obtain

$$f^{-1}f(H)\cap(X-U)=f^{-1}f(P\cap\bar{U})\cap(X-U). \tag{1}$$

The right member is a P-set so is the left member. Denoting the set in (1) above by M we obtain

$$f^{-1}f(M)\cap\bar{U}=f^{-1}f(P\cap\bar{U})\cap(\bar{U}-V).$$

The left member is a P-set so is the right member. The set $P\cap(\bar{U}-V)$ is closed for it is the intersection of the P-sets P and $f^{-1}f(P\cap\bar{U})\cap(\bar{U}-V)$. Thus X being P-generated implies $\bar{U}-V$ is a closed subset of X.

Since the closedness is admissible and every closed mapping is reflexive closed, we obtain

4.3. Corollary. *Let f be a mapping of X into Y. If f is closed then f generates an upper-semicontinuous decomposition of X.*

It is well-known the converse of Corollary 4.3 also holds.

4.4. COROLLARY. *Let X be a compactly generated space and f a mapping of X into Y. If f is reflexive compact, then f generates an upper-semicontinuous decomposition of* X [3, Theorem 1].

4.5. COROLLARY. *Let X be a paracompactly generated space and f a mapping of X into Y. If f is reflexive paracompact, then f generates an upper-semicontinuous decomposition of X.*

Combining Corollary 3.8 with Corollary 4.5 we obtain the following corollaries.

4.6. COROLLARY. *Let X be a compactly generated space and f a mapping of X into Y. If f is reflexive paracompact, then f generates an upper-semicontinuous decomposition of X.*

4.7. COROLLARY. *Let X be a locally paracompact Hausdorff space and f a mapping of X into Y. If f is reflexive paracompact, then f generates an upper-semicontinuous decomposition of X.*

References

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