TOPOLOGICAL SPACES WITH LARGE UNIFORMITIES

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The purpose of this paper is to determine sufficient conditions on the topological space $(X, \mathscr{T})$ so that $\mathscr{U}$, the neighborhood system of the diagonal Δ in $X \times X$ with the product topology, is a uniformity of X and $\mathscr{T}$ is the uniform topology of $\mathscr{N}$. We prove this

THEOREM *If $(X, \mathscr{T})$ is locally compact, T_2, and σ-compact, then $\mathscr{N}$ is a uniformity for X and $\mathscr{T}$ is the uniform topology of $\mathscr{N}$.*

These results are known:

A. If $(X, \mathscr{U})$ is a compact uniform space, then $\mathscr{N} \subset \mathscr{U}$.

B. If $(Y, \mathscr{V})$ is a uniform subspace of the space $(X, \mathscr{U})$, then the topology of the relative uniformity $\mathscr{V}$ is the relativized topology of $\mathscr{U}$.

C. X is σ-compact if $X = \bigcup_{i=1}^{\infty} X_i$ where X_i is compact for each i. If X is locally compact, T_2, and σ-compact, then $X = \bigcup_{i=1}^{\infty} X_i$ where X_i is compact and $X_i \subset X_{i+1}^{O}$ for each i.

D. If $(X, \mathscr{T})$ is a completely regular space, then there is a largest uniformity $\mathscr{V}$ for X whose uniform topology is $\mathscr{T}$.

PROOF OF THE THEOREM

It is sufficient to show that if $U \in \mathscr{N}$ then there exists $V \in \mathscr{N}$ such that $V \circ V \subset U$. Let $X = \bigcup_{i=1}^{\infty} X_i$ where X_i is compact and $X_i \subset X^{O}_{i+1}$ for each i. Since X_i is compact for each i, there exists for each i a neighborhood V'_i of $\{(x, x) : x \in X_i\}$ such that $V'_i \subset X_i \times X_i$ and $V'_i \circ V'_i \subset U$. Note that V'_i for each i is a neighborhood of $\{(x, x) : x \in X_i^{O}\}$ in $X \times X$ with the product topology. Let $V_1 = V'_1 \cap V'_2 \cap V'_3$ and $V_2 = V_1 \cup (V'_2 \cap V'_3)$. For each $X \in X_3 - X_2^{O}$ x is not a limit point of X_1; so that there exists a neighborhood N_x of x such that $N_x \cap X_1 = \phi$. Define

$$V_3 = V_1 \cup V_2 \cup \{[\cup (N_x \times N_x : x \in X_3 - X_2^{O}) \cup V_2] \cup V'_3 \cup V'_4\}$$

Suppose V_{n-1} has been constructed. For each $x \in X_n - X_{n-1}^{O}$ there exists a neighborhood N_x of x such that $N_x \cap X_{n-2} = \phi$. Define

$$V_n = \left(\bigcup_{i=1}^{n-1} V_i\right) \cup \{[\cup (N_x \times N_x : x \in X_n - X_{n-1}^{O}) \cup V_{n-1}] \cap V'_n \cap V'_{n+1}\}$$

Let $V=\bigcup_{n=1}^{\infty} V_n$. By construction $V\epsilon\mathscr{U}$. Also, $V_n\subset X_n\times X_n$ and $V_n\subset V_{n+1}$ for each n.

To show that $V\circ V\subset U$ it is sufficient to show that $V_n\circ V_n\subset U$ for each n. It is clear that $V_1\circ V_1\subset U$ and $V_2\circ V_2\subset U$. Suppose $(y,z)\epsilon V_n$ and $(z,w)\epsilon V_n$ where $n\geq 3$.

Let r be the smallest integer such that $(y,z)\epsilon V_r$, and let s be the smallest integer such that $(z,w)\epsilon V_s$.

CASE 1: $r=s\geq 3$. Then (y,z) and (z,w) are both in

$$[\bigcup(N_x\times N_x: x\epsilon X_r-X^O_{r-1})\cup V_{r-1}]\cap V'_r\cap V'_{r+1}$$

and therefore in V_r so that $(y,w)\epsilon U$

CASE 2: $3\leq s$ and $r<s$

A. $(s-r)\geq 2$. Then (z,w) is in

$$[\bigcup(N_x\times N_x: x\epsilon X_s-X^O_{s-1})\cup V_{s-1}]\cap V'_s\cap V'_{s+1}$$

so that $(z,w)\epsilon N_x\times N_x$ for some $x\epsilon X_s-X^O_{s-1}$. But $N_x\cap X_{s-2}=\phi$. so that $z\notin X_{s-2}$ and hence $z\notin X_r$ which is a contradiction since $(y,z)\epsilon V_r$ implies that $z\epsilon X_r$.

B. $r+1=s$. Then (y,z) and (z,w) are in V'_s so that $(y,w)\epsilon U$.

CASE 3: $3\geq r$ and $s<r$

Proof is similar to proof of CASE 2.

That $\mathscr{T}$ is the uniform topology of the uniformity $\mathscr{U}$ is easily established.

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