A CHARACTERIZATION OF COMPLETENESS

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It is well known that a function f on a topological space X to a topological space Y is continuous if, and only if, for each net $\{S_n, n \in D\}$ in X which converges to a point s, the composition $\{f \circ S_n, n \in D\}$ converges to $f(s)$. We also know that if f is a uniformly continuous function on a uniform space $(X, \mathscr{U})$ with values in a uniform space $(Y, \mathscr{V})$, then for each Cauchy net $\{S_n, n \in D\}$ in $(X, \mathscr{U})$, $\{f \circ S_n, n \in D\}$ is also a Cauchy net in $(Y, \mathscr{V})$. But the converse does not hold as shown by the existence of continuous but not uniformly continuous functions on a complete uniform space to a complete uniform space. However we have the following theorem.

THEOREM 1. *If f is a function on a uniform space $(X, \mathscr{U})$ with values in a uniform space $(Y, \mathscr{V})$ such that for each Cauchy net $\{S_n, n \in D\}$ in $(X, \mathscr{U})$ $\{f \circ S_n, n \in D\}$ is also a Cauchy net in $(Y, \mathscr{V})$, then f is continuous relative to the uniform topologies.*

PROOF. Let $\{S_n, n \in D\}$ be a net in $(X, \mathscr{U})$ which converges to a point s but $\{f \circ S_n, n \in D\}$ does not converge to $f(s)$. Then there exists an open neighborhood N of $f(s)$ such that $\{f \circ S_n, n \in D\}$ is frequently in the complement of N. Hence $\{S_n, n \in E\}$ is a subnet of $\{S_n, n \in D\}$ where $E = \{n \in D: f \circ S_n \notin N\}$. Let $\{T(m, n), (m, n) \in E \times E\}$ be the net in $(X, \mathscr{U})$ defined by

$$T(m, n) = s \qquad (\text{when } m \neq n)$$

and

$$T(m, m) = S_m.$$

Then $\{T(m, n), (m, n) \in E \times E\}$ converges to s and hence is a Cauchy net. Since N is a neighborhood of $f(s)$, there exists V in $\mathscr{V}$ such that $V[f(s)] \subset N$ and hence $f \circ T(m, m) \notin V[f(s)]$ for all m in E; that is, $(f \circ T(m, m), f \circ T(n, 1)) \notin V$ for all m and $n \neq 1$. Therefore $\{f \circ T(m, n), (m, n) \in E \times E\}$ is not a Cauchy net.

Since Cauchy nets and Cauchy filters are so closely related, the following theorem can be expected.

THEOREM 2. *A function f from a uniform space $(X, \mathscr{U})$ to a uniform space $(Y, \mathscr{V})$ preserves Cauchy nets if, and only if f preserves Cauchy filter bases.*

PROOF. Suppose that $\mathscr{F}$ is a Cauchy filter base in $(X, \mathscr{U})$ but $\{f(A): A \in \mathscr{F}\}$ is not a Cauchy filter base in $(Y, \mathscr{V})$, Then there exists V in $\mathscr{V}$ such that for each

A in $\mathscr{F}$, there are a_1 and a_2 in A such that $(f(a_1), f(a_2))$ is not in V. Direct $\mathscr{F} \times \{1, 2\}$ by $\geq$ as follows: $(F, m) \geq (G, n)$ if $F \subset G$ and define the function $S: \mathscr{F} \times \{1, 2\} = X$ by

$$S(A, 1) = a_1, \quad S(A, 2) = a_2$$

where a_1 and a_2 are in A and $(f(a_1), f(a_2))$ is not in V. Then $\{S(A, m), (A, m) \in \mathscr{F} \times \{1, 2\}\}$ is a Cauchy net but $\{f \circ S(A, m), (A, m) \in \mathscr{F} \times \{1, 2\}\}$ is not a Cauchy net.

Conversely let $\{S_n, n \in D\}$ be a Cauchy net and let $A_n = \{S_m: m \geq n\}$ and $\mathscr{F} = \{S_n: n \in D\}$; then $\mathscr{F}$ is a Cauchy filter base. Hence for each V in $\mathscr{V}$ there exists A_n in $\mathscr{F}$ such that $\{(f(a), f(b)): a$ and b are in $A_n\} \subset V$. Therefore $\{(f \circ S_i, f \circ S_j) : (i, j) \geq (n, n)\} \subset V$ and $\{f \circ S_m, m \in D\}$ is a Cauchy net.

If $(X, \mathscr{U})$ is a complete uniform space, it is clear that every continuous mapping on $(X, \mathscr{U})$ preserves Cauchy nets. However it is interesting that the converse is also true.

THEOREM 3. *Every continuous function f on the uniform space $(X, \mathscr{U})$ preserves Cauchy nets, if and only if $(X, \mathscr{U})$ is complete.*

PROOF. Suppose that $(X, \mathscr{U})$ is not complete and let (X^*, u^*) be a completion of $(X, \mathscr{U})$ and identify X with the image under embedding. Then there exists y in $X^* - X$ and p in the gage of (X^*, u^*) such that $p(x, y) \neq 0$ for all x in X. Define a function f on X to the reals with the usual uniformity by $f(x) = 1/p(x, y)$. Then f is continuous but does not preserve Cauchy nets.

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REFERENCES

[1] N. Bourbaki; *Topologie generale*, Actualtes Sci. Ind. Paris.
[2] J. L. Kelley; *Geneal Topology*, Van Nostrand Company, New York, 1955.
[3] J. R. Isbell; *Uniform Spaces*, Amer. Math. Soc., Providence, Rhode Island, 1964.