On the Consistency and Independence of a System of Axioms of the Propositional Calculus

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Here will be presented the consistency and independence of a system of axioms of the propositional calculus.

1. The System of Axioms. The axioms of the propositional calculus consists of primitive logical formulas (axioms) and primitive rules for inferring true formulas.

The primitive ligical formulas are the following four.

(a) $p \vee p \longrightarrow p$

(b) $p \longrightarrow p \vee q$

(c) $p \vee q \longrightarrow q \vee p$

(d) $p \longrightarrow q : \longrightarrow : r \vee p . \longrightarrow . r \vee q$

The primitive rules for inference are the following two.

(α) Rule of Substitution

Any propositional combination may be substituted for a propositional variable, provided that the substitution is made for every occurrence of the propositional variable.

(β) Modus Ponens

From the two formulas A and $A \longrightarrow B$. we get the new formula B.

NOTE: (1) Here propositional variables are designated by p, q, r,......, and by A, B,...we designate formulas *i. e.* any meaningful combinations of propositional variables.

(2) In the system of axioms we take "$-$" (negation) and "$\vee$" (disjuntion) as the sole logical connectives and "$p \longrightarrow q$" (implication) will be counted as the abbreviation of "$\bar{p} \vee q$"

(3) Dots notations as in axiom (d) will be the customary ones. That is they are used instead of parentheses.

2. The Consistency of the System of Axioms. By the consistency of the system of axioms we mean that thereis no formula A which is provable together with its negation $\overline{A}$.

To prove the consistency we proceed in the following way.

We let the sentential variables p, q, r,... take the numerical values 0 and 1 and we define "$n \vee m$" as "$n \times m$" where n and m are 0 or 1. Further we let

$$\overline{0}=1, \quad \overline{1}=0.$$

Then every formula represents an arithmetical function which assumes only the values 0 and 1.

Now we prove that every provable formula of the calculus takes the value 0 only for possible assignment of values to the propositional variables.

The proof runs as follows:

The fact that Axioms (a) through (d) possess this property is shown as follows.

It is easily tested that $\bar{p}\vee p$ always takes the value 0. Therefore $\overline{p\vee p}\vee p$, Axiom (a), takes the value 0 only, because $p\vee p$ has the same value as p. $\bar{p}\vee . p\vee q$, Axiom (b), always has the same value as $\bar{p}\vee p . \vee q$, because the arithmetical product is associative. Therefore it takes the value 0 only, because $0\vee q$ is equal to 0. Since $q\vee p$ always has the same value as $p\vee q$, $\overline{p\vee q}\vee . q\vee p$, Axiom (c), takes 0 only because it is a case of $\bar{p}\vee p$. It is easily seen that $\overline{\bar{p}\vee q}\vee : \overline{r\vee p} . \vee r\vee q$, Axiom (d), takes the value 0 only when $r=0$ because one factor becomes 0. When $r=1$ it reduce sto $\overline{\bar{p}\vee q}\vee . \bar{p}\vee q$ and this is a case of $\bar{p}\vee p$, too.

Thus the four formulas of the axioms take the value 0 only.

Now under applications of the two rules for inference this property persists. It is trivial that this is true for the Rule of Substitution because any combination of propositional variables takes the value 0 or 1 only. Now if A takes 0 and $\bar{A}\vee B$ takes 0, then B has to take the value 0 too. That is, under the application of Modus Ponens the property persists.

Thus we see that every provable formula takes the value 0 only. But if A and $\bar{A}$ were both provable in this system, both of them have to take the value 0 only, which is obviously contradictory.

3. The Independence of the Systen of Axioms. To prove the independence of Axiom (a) from the rest of axioms, we let the variables p, q, r,... take the residues 0, 1 and 2 modulo 4. The sign "$\vee$" will be taken as the ordinary multiplication, and further we define

$$\bar{0}=1,\quad \bar{1}=0,\quad \bar{2}=2.$$

Then just as in the proof of the consistency we can easily show that Axioms (b), (c) and (d) always yield the residue 0, and this property persists under the applications of the two rules for inference.

Therefore if $\overline{p\vee p}\vee p$, Axiom (a), were deducible from (b), (c) and (d) it would yield the residue 0 only. But when we let p take the value 2 Axiom (a) takes the value 2.

To prove the independence of $\bar{p}\vee . p\vee q$, Axiom (b), from the rest of the axioms, we let p, q, r, ... take the value 0, 1, 2, and 3. And we define "$m\vee n$" to be the minimum of m and n and further we define

$$\bar{0}=1,\quad \bar{1}=0,\quad \bar{2}=3,\quad \bar{3}=2.$$

Then Axioms (a), (c) and (d) always take the value 0 or 2. And this property persists under the applications of the two rules for inference. But when we let $p=2$ and $q=1$, $\bar{p}\vee . q\vee q$ becomes 1.

To show the independence of Axiom (c) $\overline{p\vee q}\vee . q\vee p$ again we let the variables take 0, 1, 2 and 3 and

$$\bar{0}=1,\quad \bar{1}=0,\quad \bar{2}=0,\quad \bar{3}=2.$$

Further when m or n is 0 or 1 "$m \vee n$" will be the ordinary multiplication and

$$2 \vee 3=0, \ 3 \vee 2=3, \ 2 \vee 2=2, \ 3 \vee 3=3.$$

Then it is seen that Axioms (a), (b) and (d) always take the value 0 and this property persists under the applications of the two rules for inference.

But when $p=2$ and $q=3$ Axiom (c) takes the value 3.

Finally to show the independence of Axiom (d) from the rest of the axioms again we let the variables take the values 0, 1, 2 and 3. And let

$$\overline{0}=1, \ \overline{1}=0, \ \overline{2}=3, \ \overline{3}=0$$

and except

$$2 \vee 2=2, \ 2 \vee 3=3 \vee 2=0, \ 3 \vee 3=3$$

"$n \vee m$" will be the ordinary multiplication.

Then axioms (a), (b) and (c) always take the value 0 and this property persists under the applications of the two rules for inference. However when $p=3$, $q=1$ and $r=2$, Axiom (d) takes the value 2.

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