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TRIGONOMETRY IN A HYPERBOLIC SPACE

BY

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1. Introduction. We consider a non-euclidean triangle in the Kähler maniffold (H, M) furnished with the Bergman metric $M(H)$ on the hypersphere H in the complex euclidean space C^n of complex dimension n. (Throughout this paper n stands for any integer ≥ 2.) By a non-euclidean triangle we mean a geodesic triangle with the ordinary angle replaced by the analytic angle (see §2 for definition).

The main object of this paper is to study on a given non-euclidean triangle in (H, M) some basic trigonometric identities, such as the laws of sines and cosines which arise in hyperbolic geometry, and related problems. For a non-euclidean triangle in (H, M) the law of sines holds while the law of cosines may not. The law of cosines holds if and only if one of the three angles of the non-euclidean triangle is equal to the corresponding ordinary angle (see §3). From this follows immediately a stronger form of Pythagorean theorem obtained in [2]. As was pointed out in [2] the following statement is not true in general: Through a point c in H not lying on a given geodesic γ_H at the right analytic angle. Theorem 4.1 gives such an example. A necessary and sufficient criterion that the above statement be true is given in Theorem 4.2. There are domains D in which the theorems in §3 fail to hold in (D, M). Such examples are exhibited in §5.

2. Preliminaries. We consider a bounded domain D in the complex euclidean space C^n of complex dimension n with the coordinates $z=(z^1, \cdots\cdots, z^n)$. The Bergman metric

$$M(D): \quad ds_D{}^2=T_{\alpha\bar{\beta}}\, dz^\alpha\, d\bar{z}^\beta$$

of D is kählerian and invariant under biholomorphic mappings of D[1]; here we use the summation convention.

For any tangent vectors $u=(u^\alpha)$ and $v=(v^\alpha)$ at z in the Kähler manifold (D, M) we let

$$(2.1) \qquad [u, v]=T_{\alpha\bar{\beta}}u^\alpha\bar{v}^\beta, \quad \|u\|^2=[u, u].$$

If two vectors u and v are independent in $C^n(R)$, these vectors determine a plane section, $S(u, v)=[w=\lambda u+\mu v]$, where λ, μ are real parameters. Let $S_z(u, v)$ be a section defined by two tangent vectors u and v at z in D. It is called *real* if $Im[u, v]=0$ at z and *analytic* if $v=\lambda u$ for some $\lambda\in C$. The ordinary angle Θ between two vectors u and v at $z\in D$ is given by

$$(2.2) \qquad \cos\Theta=|[u, v]|\,/\,\|u\|\ \|v\|.$$

Besides the ordinary angle we use the notion of anlytic angle. It is given by

$$(2.3) \qquad \cos F=|[u, v]|\,/\,\|u\|\ \|v\|, \quad 0\leq F\leq\pi/2.$$

We observe that (1°) $\Theta\geq F$ and the equality holds if and, only if the section $S_z(u, v)$ is seal and (2°) $F=0$ if and only if $S_z(u, v)$ is analytic. Since the metric $M(D)$ is invariant, all geometric quantities defined on (D, M) are invariant under biholomorphic mappings of D.

We remark that the formulas from (2.1) to (2.3) may be used to define the corresponding quantities in the euclidean measure if the metric tensor $T_{\alpha\bar\beta}$ is replaced by $\delta_{\alpha\bar\beta}$.

We now consider the hypersphere

$$H=[z:\ |z|<1],\quad |z|^2=(z, z).$$

The Bergman metric of H is

$$M(H):\ ds_H^2(z)=(n+1)(1-|z|^2\sin^2 f)(1-|z|^2)^{-2}|dz|^2,$$

where f is the euclidean analytic angle between two vectors z and dz (see also [2] for definition). Let a and b be two points in H. The holomorphic automorphism of H which maps b into the origin o is

$$\begin{aligned} &w=w(z)=q(z-b)(I-b^*z)^{-1}r^{-1}, \\ &q\bar{q}=(1-bb^*)^{-1},\ r^*r-(I-b^*b)^{-1},\ |b|<1 \end{aligned} \tag{2.4}$$

and I is the $n\times n$ identity matrix and b^* denotes the complex conjugate transpose of b. Under this mapping the point a is mapped into the point

$$\begin{aligned} &w(a)=A=q(a-b)(I-b^*a)^{-1}r^{-1}, \\ &|A|^2=(A, A)=(|a-b|^2-\boldsymbol{b}^2(P)\ /\ |1-(a, b)|^2 \end{aligned} \tag{2.5}$$

where $\boldsymbol{b}(P)$ is the B-area of the parallelogram P determined by a, b and the origin (see [2]).

3. The laws of sines and cosines. For any pair of points a and b in H there is a unique shortest geodesic $\gamma_H(a, b)$ connecting a and b, and it is given by the image curve of the straight line segment connecting the origin with A, the image of a under (2.4), under the inverse mapping $z=z(w)$ of (2.4). Hence three points a, b and c in H determine a non-euclidean triangle $\triangle$ uniquely.

THEOREM 3.1. *Let $\triangle$ be a non-euclidean triangle in (H, M) whose three (non-euclidean analytic) angles are F_1, F_2 and F_3. Let g_i be the non-euclidean lengths of three sides of $\triangle$ opposite to F_i, $i=1, 2, 3$. Then*

$$\frac{\sin F_1}{\sinh\frac{g_1}{(n+1)^{\frac{1}{2}}}}=\frac{\sin F_2}{\sinh\frac{g_2}{(n+1)^{\frac{1}{2}}}}=\frac{\sin F_3}{\sinh\frac{g_3}{(n+1)^{\frac{1}{2}}}} \tag{3.1}$$

Proof. First we consider a non-euclidean triangle $\triangle$ of vertices a, b and the origin o. Let F_1, F_2 and F_3 be the angles of $\triangle$ at the vertices a, b and o, respectively, and f_i the corresponding angles of F_i in the euclidean measure. By definition we have

$$\begin{aligned} &\cos f_1=|(b-a, a)|\ /\ |a-b|\,|a| \\ &\cos f_2=|(a-b, b)|\ /\ |b|\,|a-b| \\ &\cos f_3=|(a, b)|\ /\ |a|\,|b|. \end{aligned} \tag{3.2}$$

From Lemma 3.1 in [2] it can be shown by a formal computation that

$$\begin{aligned} &\sin^2 F_1=(1-|a|^2)\boldsymbol{b}^2(P)\ /\ |a|^2(|a-b|^2-\boldsymbol{b}^2(P)) \\ &\sin^2 F_2=(1-|b|^2)\boldsymbol{b}^2(P)\ /\ |b|^2(|a-b|^2-\boldsymbol{b}^2(P)) \\ &\sin^2 F_3=\boldsymbol{b}^2(P)\ /\ |a|^2|b|^2, \end{aligned} \tag{3.3}$$

where $b(P)$ denotes the B-area of the parallelogram P determined by a, b and o. From Lemma 4.1 in [2] we have

$$\sinh^2 \frac{g_3}{(n+1)^{\frac{1}{2}}} = |A|^2 / (1-|A|^2), \tag{3.4}$$

where A is given in (2.5). Since $1-|A|^2 = \frac{(1-|a|^2)(1-|b|^2)}{|1-(a,b)|^2}$, we obtain

$$\sinh^2 \frac{g_3}{(n+1)^{\frac{1}{2}}} = (|a-b^2|-b^2(P)) / (1-|a|^2)(1-|b|^2). \tag{3.5}$$

In particular,

$$\sinh^2 \frac{g_1}{(n+1)^{\frac{1}{2}}} = |b|^2 / (1-|b|^2)$$

$$\sinh^2 \frac{g_2}{(n+1)^{\frac{1}{2}}} = |a|^2 / (1-|a|^2). \tag{3.6}$$

The theorem follows from (3.3), (3.5) and (3.6) if one of the vertices, say c, is the origin. If c is not the origin, the mapping

$$w=w(z)=q(z-c)(I-c^*z)^{-1}r^{-1},$$

$$q\bar{q}=(1-cc^*)^{-1}, \quad r^*r=(I-c^*c)^{-1}, \quad |c|<1, \tag{3.7}$$

maps c into the origin. Let A and B be the respective images of a and b under this mapping. Since $g_1=g(o,B)=g(b,c)$, $g_2=g(o,A)=g(a,c)$, $g_3=g(A,B)=g(a,b)$ and the angles are invariant, the theorem now follows. Here $g(a,b)$ denotes the non-euclidean length of the shortest geodesic joining a and b.

THEOREM 3.2. *Let $\triangle$ be a non-euclidean triangle in (H, M) with the same notations as in Theorem 3.1. Then*

$$\cosh \frac{g_3}{(n+1)^{\frac{1}{2}}} = \cosh \frac{g_1}{(n+1)^{\frac{1}{2}}} \cosh \frac{g_2}{(n+1)^{\frac{1}{2}}} - \sinh \frac{g_1}{(n+1)^{\frac{1}{2}}} \sinh \frac{g_2}{(n+1)^{\frac{1}{2}}} \cos F_3 \tag{3.7}$$

if and only if $F_3=\Theta_3$, where Θ_3 is the corresponding ordinary angle of F_3.

Proof. In view of the proof of Theorem 3.1 it suffices to prove the theorem for the case where one of three vertices a, b and c, say c, is the origin o. From (3.4) we obtain

$$\cosh \frac{g_3}{(n+1)^{\frac{1}{2}}} = |1-(a,b)| / (1-|a|^2)^{\frac{1}{2}}(1-|b|^2)^{\frac{1}{2}}. \tag{3.8}$$

In particular, $\cosh \frac{g_1}{(n+1)^{\frac{1}{2}}} = 1/(1-|b|^2)^{\frac{1}{2}}$, $\cosh \frac{g_2}{(n+1)^{\frac{1}{2}}} = 1/(1-|a|^2)^{\frac{1}{2}}$.

From (3.3) we also have $\cos F_3 = |(a,b)| / |a||b|$. It can be easily checked that (3.7) holds if and only if $Im\,(a,b)=0$ or $F_3=\Theta_3$ at the origin.

In particuar, if $F_3=90°$ in Theorem 3.2 we have the following stronger form of the non-euclidean Pythagorean theorem:

THEOREM 3.3. *Let $\triangle$ be a non-euclidean triangle in (H, M) with the same notations as in Theorem* 3.1. *Then*

$$\cosh \frac{g_3}{(n+1)^{\frac{1}{2}}} = \cosh \frac{g_1}{(n+1)^{\frac{1}{2}}} = \cosh \frac{g_2}{(n+1)^{\frac{1}{2}}}$$

holds if and only if $F_3=90°$. (Compare with Theorem 4.2 in [2]).

Since the metric $M(H)$ is invariant the above theorems hold for any domain which is biholomorphically equivalent to the hypersphere, in particular, for any bounded domain furnished with the complete Bergman metric with constant sectional curvature. The second part follows from the result due to K. H. Look [6] which states that any bounded domain D furnished with the complete Bergman metric with constand sectional curvature is biholomorphically equivalent to the hypersphere.

4. Perpendiculars in the Space (H, M). As was pointed out in [2] the following statement is not true in general: Through a point in (H, M) not lying on a given geodesic γ_H there exists at least one geodesic intersecting γ_H at the right analytic angle. It may be shown as follows: Since the geodesic $\gamma_H(a, b)$ joining two points a and b in (H, M) is the image curve of the straight line oA, where A is given in (2.5), under the inverse mapping

(4.1) $$z=z(w)=((1-|b|^2)^{\frac{1}{2}}wr+b) / (1-(1-|b|^2)^{\frac{1}{2}}wrb^*)$$

of (2.4), the parametric equation of $\gamma_H(a, b)$ is given by

(4.2) $$z=(Pt+b) / (1+(P, b)t), \quad 0 \leq t \leq 1,$$

where

(4.3) $$P=(p^1, \cdots\cdots, p^n)=(a-b)(I-b^*a)^{-1}$$

A formal computation shows that

(4.4) $$p^j=a^j(1-|b|^2) / (1-(a, b))-b^j$$

and hence, $z^j / z^p=a^j / a^p$ for all t, $0 \leq t \leq 1$, if $b^j/b^p=a^j/a^p$ for $j \neq p$. From this we obtain

LEMMA 4.1. *Let a and b bd two points in H which determine a plane section $S(a, b)$. If $S(a, b)$ is anaytic then the geodesic $\gamma_H(a, b)$ lies completely on the section $S(a, b)$.*

The following theorem follows immediately from Lemma 4.1.

THEOREM 4.1. *Through a point in H not lying on a given geodesic γ_H there may not exist a geodesic which intersects γ_H at the right analytic angle.*

Further we have

THEOREM 4.2. *Through a point c in H not lying on a given geodesic γ_H there exists a geodesic which intersects γ_H at the right analytic angle if and only if there are two distinct points a and b on γ_H such that the the two tangent vectors at c along $\gamma_H(a, b)$* and $\gamma_H(b, c)$ *form a real section at c.*

Proof. Frist we take c to be the origin o and γ_H to be any geodesic which does not pass through the origin. Let a, b be two distinct points on γ_H. The equation of γ_H is then given by (4.2) with a real parameter t. A straightforward computation yields

(4.1) $$dz / dt=P(I-b^*b) / (1+Pb^*t)^2$$

and

(4.6) $$[dz/dt,\ z]=(n+1)(dz/dt,\ z)/(1-|z|^2)^2.$$

We look for the real solution t of the equation:

(4.7) $$(dz/dt,\ z)=0$$

or

$$(P(I-b^*b),\ Pt+b)=0$$

From $1-bb^*>0$, we have $I-b^*b>0$ (see p. 33, [4]). Hence $P(I-b^*b)P^*=(P-Pb^*b)P^*=|P|^2-|Pb^*|^2>0$, and the solution of (4.7) is given by

$$t=-Pb^*(1-|b|^2)/(|P|^2-|Pb^*|^2).$$

The solution t is real if and only if $Im\ (P,\ b)=0$. A formal computation shows that $Im(P,\ b)=Im(a,\ b)(1-|b|^2)$. Hence there exists a geodesic through the origin o which intersects γ_H at the right analytic angle if and only if $Im[a,\ b]=Im(a,\ b)=0$. If c is not the origin, applying the mapping (3.7) in H which maps c into the origin we can reduce to the above case.

5. Examples. In this section we exhibit domains D in which the theorems in §3 or similar theorems fail to be true in $(D,\ M)$. Such domains arise among the class of domains which are not biholomorphically equivalent to the hypersphere H. This class includes simple domains such as the polydisc $P=[z:|z^j|<1,\ j=1,\ 2,\cdots\cdots,\ n]$ and the classical Cartan domain of type I: $R=[Z: I-ZZ^*>0]$, where Z is an $n\times m$ matrix of complex elements $(n\geq m\geq 2)$, Z^* its conjugate transpose and I the identity matrix of order n. P is a reducible symmetric domain while R is an irreducible symmetric domain. It is known that

$$M(P):\ ds_P^2(z)T_{\alpha\bar{\alpha}}|dz^\alpha|^2=2\sum_{\alpha=1}^{n}(1-|z^\alpha|^2)^{-2}|dz^\alpha|$$

and

$$M(R):\ ds_R^2(Z)=\sigma((I-ZZ^*)^{-1}dZ(I-Z^*Z)^{-1}dZ^*)\ \text{(see [3] or [7])},$$

where σ denotes the trace of the matrix. Furthermore, if a and b are two points on P then

(5.1) $$g_P(a,b)=2^{-\frac{1}{2}}\Big[\sum_{j=1}^{n}\log^2\frac{1+|Q_j|}{1-|Q_j|}\Big]^{\frac{1}{2}},$$

where $Q_k=(b^j-a^j)/(1-a^{-j}b^j)$, and

(5.2) $$g_R(A,\ B)=\Big(\frac{n+m}{4}\Big)^{\frac{1}{2}}\Big[\sum_{j=1}^{m}\log^2\frac{1+\lambda_j}{1-\lambda_j}\Big]^{\frac{1}{2}}$$

for any two points A and B in R, where λ_j are the positive square roots of the characteristic roots of the matrix

(5.3) $$(B-A)(I-A^*B)^{-1}(B^*-A^*)(I-AB^*)^{-1}$$

and $1>\lambda_1\geq\cdots\cdots\geq\lambda_m\geq 0$ [3].

Example 1. The bicylinder $B=[z:|z^\alpha|<1,\ \alpha=1,\ 2]$ in the space C^2 and the points $a=\Big(\frac{1}{2},\frac{1}{3}\Big)$, $b=\Big(-\frac{1}{3},\frac{1}{2}\Big)$ and $o=(0,\ 0)$ in B. At the origin we have $T_{\alpha\bar{\alpha}}=2$ for $\alpha=1,\ 2$. Hence angles in both the euclidean measure and non-euclidean measure coincide at the origin. Let F_1, F_2, and F_3 denote the non-euclidean analytic angles of the triangle abo at a, b and o, respectivly. Then $\cos F_3=\cos f_3=|(a,\ b)|/|a||b|=0$. Since the holomorphic automorphism.

(5.4) $$w^\alpha=(z^\alpha-a^\alpha)/(1-a^{-\alpha}z^\alpha),\ \alpha=1,\ 2.$$

of B maps a, b and o into $w(a)=o$, $w(b)$ and $w(o)=-a$, respectively, $\cos F_1=|(w(b), w(o))|/|w(b)||w(o)|=61/(8762)^{\frac{1}{2}}$. Similarly, $\cos F_2=71/(8762)^{\frac{1}{2}}$. Further, form (5.1) we have $g_B(a,b)=(\log^2 2+\log^2 3)^{\frac{1}{2}}$, $g_B(a,o)=g_B(o,b)=2^{-\frac{1}{2}}(\log^2 2+\log^2 3)^{\frac{1}{2}}$. A numerical computation now shows that Theorems 3.1–3.3 fail to be true in this case.

Example 2. The classical Cartan domain R of type I for $n=m=2$ and the points

$$A=\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}, \quad B=\begin{pmatrix} -\frac{1}{3} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

in R. Let F_1, F_2 and F_3 denote the non-euclidean analytic angles of the triangle ABo at A, B and o, respectively. From the metric $M(R)$ we see that

(5.5) $$\cos F_3=\cos f_3=|\sigma(AB^*)|/\sigma(AA^*)^{\frac{1}{2}}\sigma(BB^*)^{\frac{1}{2}}$$

at the origin o and, hence, $\cos F_3=0$. The holomorphic automorphism

(5.6) $$W=Q(Z-A)(I-A^*Z)^{-1}R^{-1}, \quad I-AA^*=(Q^*Q)^{-1}=(RR^*)^{-1}\ [3]$$

of R maps A, B and o into $W(A)=o$, $W(B)$ and $W(o)=-QAR^{-1}$.

It is easy to see that $Q=R=\begin{pmatrix} 2/\sqrt{3} & 0 \\ 0 & 3/(\sqrt{2})^3 \end{pmatrix}$, and hence,

$$W(o)=-\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} \text{ and } W(B)=\begin{pmatrix} -\frac{5}{7} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}.$$

A formal computation shows that $\cos F_1=61/(8762)^{\frac{1}{2}}$. Similarly, $\cos F_2=71/(8762)^{\frac{1}{2}}$. Further, from (6.2) and (5.3) we obtain that $g_R(A,B)=2^{\frac{1}{2}}(\log^2 2+\log 3)^{\frac{1}{2}}$, $g_R(A,o)=(\log^2 2+\log^2 3)^{\frac{1}{2}}=g_R(o,B)$. As before this shows that the theorems in §3 are not true in this case.

It would be of interest to investigate whether or not the theorems in §3 or similar theorems should hold in a symmetric domain which is not a biholomorphic image of the hypersphere when it is furnished with a suitable metric.

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