Vol. 4, No. 1, 1967. 10.

SEQUENTIALLY COMPLETE UNIFORM SPACES WHICH ARE NOT COMPLETE

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Introduction. In this thesis, we are concerned with conditions under which a topological space admits an incomplete uniform structure that makes every Cauchy sequence to converge. Any such condition must include noncompactness, since every compact uniform space is complete. On the other hand, counter examples show that one can not exclude first countable spaces despite the fact that each limit point is a sequential limit. Rather, the problem seems to have little to do with countability axioms as Theorem 3 below says that every noncompact normal Hausdorff space admits a sequentially complete but incomplete uniform structure.

Section 1 is devoted to giving a complete characterization of countabe absolute G_δ-sets. By using this characterization, we show in Section 2 that every noncompact, countable, absolute G_δ-set admits an incomplete uniform structure making Cauchy sequence convergent. In the final section, this result is further generalized to noncompact normal Hausdorff spaces.

It is readily seen that if a space X admits a sequentially complete uniform structure $\mathscr{U}$, then X is sequentially complete with respect to any uniform structure larger than $\mathscr{U}$. Hence, if a topologically complete space admits an incomplete, sequentially complete uniform structure $\mathscr{U}$, it should be interesting to know whether $\mathscr{U}$ can be enlarged without breaking incompleteness. Since the incomplete structure described in Lemma 3 is evidently larger than those given by Chang in [1] and [2], Theorem 2 may be of some interest in this sense although it is an immedidiate consequence of Theorem 3.

Before proceeding to prove the result, we must have for convenience a brief review of necessary definitions. By a ***uniform structure*** on a set X is meant any novoid collection $\mathscr{U}$ of subsets of $X\times X$ containing the diagonal $\Delta=\{(x,x) : x\in X\}$ such that (1) $\mathscr{U}$ is closed under the formation of finite intersections and super sets, (2) $U^{-1}\in\mathscr{U}$ if $U\in\mathscr{U}$, and (3) $U\in\mathscr{U}$ then there is $V\in\mathscr{U}$ with $V\cdot V\subset U$. The set X equipped with the structure $\mathscr{U}$ is called a ***uniform space.*** Any uniform structure $\mathscr{U}$ gives in a natural way a topology for X called the ***uniform topology*** of $\mathscr{U}$, by setting that the sets $U[x]$, $U\in\mathscr{U}$, form a base at $x\in X$. A topological space is said to ***admit*** the uniform structure $\mathscr{U}$ if the uniform topology of $\mathscr{U}$ coincides with the original topology. Finally, a net S in a uniform space $(X,\mathscr{U})$ is called a ***Cauchy net*** if the product net $S\times S$ is eventually in each member of $\mathscr{U}$, and $(X,\mathscr{U})$ is said to be ***complete*** if every Cauchy net converges to a point in X. A uniform space in which Cauchy sequences are convergent may then by termed as "*sequen-*

tially complete."

1. Characterization of countable absolute G_δ-sets. Recall that a space is an ***absolute G_δ-set*** if it is metrizable and is a G_δ-set in every metric space in which it is topologically embedded. A set X in a space is *scattered* if every subset of X has an isolated point. The purpose of this section is to show that a countable metric space is an absolute G_δ-set if and only if it is scattered. Our main tools in achieving this goal are the Baire category theorem and a result of Alexandroff stating that a space is an absolute G_δ-set if and only if it is homeomorphic to a complete metric space. (See [3, Theorem 2-76] or [4, 6-K])

LEMMA 1. *Any complete metric space consisting of a countable number of points is scattered.*

Proof. Let X be a countable complete metric space and Y be a subset of X. Y is a G_δ-set in X, since $X-Y$ consists of at most a countable number of points. By [3, Theorem 2-76], Y admits a complete metric, and by the Baire category theorem, Y contains at least one isolated point.

Lemma 1 enables us to prove the following main result of this section.

THEOREM 1. *A countabbe space X is an absolute G_δ-set if and only if it is a scattered, first countable, regular T_1-space.*

Proof. Let X be a countable absolute G_δ-set, then obviously X is a first countable T_1-space. Since X admits a complete metric, X is scattered by Lemma 1.

Conversely, if X is a first countable regular T_1-space having a countable number of points, then X is a second countable regular T_1-space. Hence by the Urysohn metrization theorem, X is metrizable. Now let $X=X_0$, and let X_1 be the set of limit points of X_0. For each ordianl number $\alpha\geq 1$, define X_α as follows: Suppose that X_β are already defined for all ordinals $\beta<\alpha$, then X_α consists of those points of X which are limit points of each X_β, $\beta<\alpha$. Thus $\alpha>\beta$ implies $X_\alpha\subset X_\beta$, and scatteredness implies $X_\alpha-X_{\alpha+1}\neq\phi$ whenever $X_\alpha\neq\phi$. Since X is a countable set, there exists a countable ordinal number γ such that $X_\gamma=\phi$. Hence X is a countably discrete collection of points in the sense of [5], and X is an absolute G_δ-set by [5, Lemma 3].

COROLLARY. *A countable absolute G_δ-set embedded in a dense-in-itself metric space is nowhere dense.*

Proof. Let X be a metric space which is dense-in-itself, let Y be a countable absolute G_δ-set in X, and suppose contrary that Y is dense in some open set U. By scatteredness, $U\cap Y$ contains an isolated point p and there exists an open set V contained in U such that $V\cap Y$ consists of the single point p. Since no point of X is isolated, Y can not be dense in V contradicting to the assumption that Y be dense in U. This proves the corollary.

It should be remarked that the characterizing conditions in Theorem 1 for countable absolute G_δ-sets can not be proper[illegible]ed in the sense that both first countability and regularity are indispensable. Before proving this, however, we must make some preliminaries.

A *filter* in a set X is a nonvoid family $\mathscr{F}$ of nonempty subsets of X closed under the formation of finite intersections and super sets. A subfamily $\mathscr{B}$ of $\mathscr{F}$ is a *base* for $\mathscr{F}$ if each member of $\mathscr{F}$ contains a member of $\mathscr{B}$. It is a straight consequence of the Hausdorff maximal principle that every filter in X is contained in an *ultrafilter*, i.e., a filter which fails to be properly contained in any filter in X. Notice that if a subset F of X meets every member of an ultrafilter $\mathscr{F}$, then F itself is a member of $\mathscr{F}$. Finally, we call a filter $\mathscr{F}$ *free* or *fixed* according as the intersection of members of $\mathscr{F}$ is empty or not.

LEMMA 2. *No free ultrafilter has a countable base.*

This lemma is known. For the sake of completeness, however, we give a proof here.

Proof of Lemma 2. Let $\mathscr{F}$ be a free ultrafilter and suppose that there exists a countable base $\mathscr{B}=\{B_1, B_2, \cdots\}$ for $\mathscr{F}$. We can assume here that $B_{n+1}\subset B_n$ as otherwise we may take $B_1\cap B_2\cap\cdots\cap B_n$ instead of B_n. Here, infinitely many inclusions $B_{n+1}\subset B_n$ should be proer as otherwise $\mathscr{F}$ can not be free. Hence we may suppose that each B_n-B_{n+1} has at least two points by taking a cofinal subfamily of $\mathscr{B}$ if necessary. From each B_n-B_{n+1}, pick two different points p_n and q_n. Let $P=\{p_n : n\in N\}$ and $Q=\{q_n : n\in N\}$ where N denotes the set of natural numbers. The sets P, Q meet every member of $\mathscr{F}$ as they meet every member of $\mathscr{B}$. Therefore, $P\in\mathscr{F}$ and $Q\in\mathscr{F}$, but $P\cap Q=\phi$. This contradiction proves the lemma.

Now, we exhibit an example which shows that the first countability condition in Theorem 1 can not be dropped.

EXAMPLE 1. Let N denote the set of natural numbers and let $\mathscr{F}$ be a free ultrafilter in N. Such a filter exists since N is infinite. Topologize the set X of nonnegative integers by defining that a set is open if and only if it is a subset of N or is a member of $\mathscr{F}$ plus the number 0. It is easy to check that X becomes a scattered T_1-space with this topology. To prove that X is regular, let p be any point and let C be a closed set not containing p. If $p=0$, then $C\subset N$ so C is open. On thne other hand, the closed set C which fails to contain 0 must be of the form $X-(\{0\}\cup F)$ for some $F\in\mathscr{F}$, so $\{0\}\cup F$ and C are disjoint open sets containing 0 and C respectively. If $p\neq 0$, then the point p itself is open. Morever the set $X-\{p\}$ is also open since X is T_1. Hence the space X is regular and T_1, but X is not metrizable since it fails to have a countable base at 0 by Lemma 2.

The following counter example shows that the condition of regularity in Theorem 1 can not be replaced by "Hausdorff."

EXAMPLE 2. Let $X_0=\{{}^1/_{2^i} : i\in N\}$, let $X_n=\{{}^1/_{2^n}+{}^1/_{2^i} : i\in N \text{ and } i>n\}$ for each positive integer n, and let $X=\{0\}\cup(\cup X_n)$. We define a set A to be open if and only if $A=Q-B$, where Q is open in the usual topology and B is a subset of X_0. This defines a topology on X which makes X a T_2-space because it contains more open sets than the usual Hausdorff topology. It is also easy to verify that X is first countable and scattered. However, X is not regular as can be seen by choosing the point 0 and the closed set X_0.

2. Sequentially complete uniform structures for countable absolute G_δ-sets. One might suspect that a first countable uniform space would be compelete if every Cauchy sequence converged. Unfortunately, however, this suspicion is groundless as Kelley points out. Perhaps, the simplest counter example would be the one given in [4, 6-E]. A natural question would then be that which space should admit an incomplete uniform structure in which every Cauchy sequence converges. In [1] and [2], Chang shows that this is the case in every infinite discrete space. As a first step to generalize this result, we prove in this section that the same is true for every noncompact countable absolute G_δ-set.

We begin by giving an alternate proof of Chang's result. Namely,

LEMMA 3. *Every infinite discrete space admits an incomlpete uniform structure in which every Cauchy scquence converges.*

Proof. Let X be an infinite set, $\mathscr{F}$ be a free ultrafilter on X, and $\mathscr{B}$ be the collection consisting of the sets of the form $(F\times F)\cup\triangle$ where $F\in\mathscr{F}$ and $\triangle$ is the diagonal in $X\times X$. A straightforward check shows that $\mathscr{B}$ satisfies the condition of [4, Theorem 6-2], so $\mathscr{B}$ is a base for a uniformity $\mathscr{U}$ for X. Moreover, X with this uniform structure becomes a discrete space, because for any $x\in X$, there exists $F\in\mathscr{F}$ such that $x\notin F$, hence $U[x]=\{x\}$, $U=(F\times F)\cup\triangle$.

We observe that every Cauchy sequence relative to $\mathscr{U}$ is eventually constant. Let $S=\{x_i : i\in N\}$ be a sequence which is not eventually constant. Suppose first that S is eventually in a finite set A. There are at least two distinct points p and q in A such that S takes frequently the values p and q, for if not S must be eventually constant. Since $\mathscr{F}$ is a free filter, $F=X-\{p\}$ belongs to $\mathscr{F}$ and $U=(F\times F)\cup\triangle$ is a member of $\mathscr{U}$. However, $S\times S$ can not be eventually in U because $S\times S$ takes frequntly the value (p,q) but $(p, q)\notin U$. Thus S can not be a Cauchy sequence. Next suppose that S is not eventually in a finite set. Then the range of S must be an infinite set, and is a sum of two infinite sets A and B. Since S must be frequently in both A and B. so does it in each of A and $X-A$. Now, one of the sets A and $X-A$ belongs to $\mathscr{F}$ by the maximality of $\mathscr{F}$. If we denote this set by F, then $S\times S$ can not be eventually in $(F\times F)\cup\triangle$ and S is not a Cauchy sequence. This proves that every Cauchy sequence is eventually constant. Hence every Cauchy sequence converges.

To complete the proof, it only remains to show that $(X,\mathscr{U})$ is not complete. In fact the filter $\mathscr{F}$ used in defining our uniform structure $\mathscr{U}$ is a Cauchy filter by the very definition of $\mathscr{U}$ Thus $(X,\mathscr{U})$ can not be complete as the Cauchy filter $\mathscr{F}$ is not fixed.

From this result we can derive the following theorem.

THEOREM 2. *Every noncompact countable absolute G_δ-set admits an incomplete uniform structure in which every Cauchy sequence converges.*

Proof. Let X be a noncompact countable absolute G_δ-set, and X_1 be the set of all limit points of X. We show first that there exists an infinite subset of $X-X_1$ which fails to have a limit point in X. To do this, suppose that a metric is given in X and let $A=\{X_n\}$ be a countably infinite discrete closed subset of X_0. Such a subset A exists since X is noncompact. By discreteness, there exist positive numbers e_n, $n=1, 2, \cdots$, such that the e_n-

neighborhoods $S(x_n, e_n)$ of x_n are disjoint open sets. Snice $X-X_1$ is dense in X, we can choose, for each n, a point y_n in $S(x_n, e_n)\cap(X-X_1)$ such that the distance beween x_n and y_n is less than $1/n$. The set $B=\{y_n\}$ is discrete since its points are contained in the disjoint open sets $S(x_n, e_n)$. If B has a limit point y, then there exists y_n such that $y_n\in S(y,e/2)$, so $x_n\in S(y,e)$ for $e>0$ however small. This means, however, that y is also a limit point of A, which is an obvious contradiction. Hence B is a discrete closed subset of X_0. We observe that B is also open since it is a subset of the open discrete subset $X-X_1$.

Now, by Lemma 3, there exists an incomplete uniform structure $\mathscr{U}$ for B in which every Cauchy sequence converges. On the other hand, since $X-B$ must be an absolute G_δ-set, $X-B$ admits a complete metric uniform structure $\mathscr{V}$. Let $\mathscr{W}$ be the uniformity generated by the sets of the form $U\cup V$, $U\in\mathscr{U}$, $V\in\mathscr{V}$. By [4, Theorem 6-2] and the fact that B is open and closed, $\mathscr{W}$ is indeed an admissible uniformity for X. If a sequence S is frequently in both of B and $X-B$, then $S\times S$ is frequently in the complement of $U\cup V$ for any $U\in\mathscr{U}$, $V\in\mathscr{V}$. Hence each Cauchy sequence must be eventually in one of the sets B and $X-B$, and so every Cauchy sequence in $(X, \mathscr{W})$ converges. However, $(X, \mathscr{W})$ can not be complete since the closed subspace B in not complete.

3. Sequential completeness of normal Hausdorff spaces. We have seen that noncompact countable absolute G_δ-sets admit sequentially complete uniform structures that fail to be complete. Our final goal in the present note is to generalize this result to normal Hausdorff spaces.

The following seems generally accepted.

LEMMA 4. *Let X be a completely regular T_1-space and let C^* denote the set of real valued bounded continuous functions defined on X. Then the collection of the sets of the form $\{(x, y) : |f(x)-f(y)|<r\}$, $r>0, f\in C^*$, constitutes a subbase for an admissible precompact uniform structure for X.*

Using the above lemma we establish the fololwing theorem.

THEOREM 3. *Every normal Hausdorff space admits a precompact, sequentially complete uniform structure. Accordingly, every noncompact normal Hausdorff space admits an incomplete uniform structure in which every Cauchy sequence converges.*

Proof. Let X be a normal Hausdorff space, C^* be the set of all real valued bounded continuous functions on X, and $\mathscr{U}$ be the uniform structure generated by the sets of the form

$$U(f, r)=\{(x, y) : |f(x)-f(y)|<r\}, f\in C^*, r>0.$$

Since every normal Hausdorff space is completely regular and T_1, $\mathscr{U}$ is a well definded admissible precompact uniform structure for X by virtue of Lemma 4. In order to prove sequential completeness of $\mathscr{U}$, we show that no divergent sequence can be a Cauchy sequence. To do this, let $S=\{x_n : n\in N\}$ be a sequence that fails to converge. We have either (1) S has no cluster point, or (2) S has at least one cluster point.

Case (1). By taking a subsequence if necessary, we may suppose $x_i\neq x_j$ if $i\neq j$. Let $A=\{x_{2n-1}\}$, $B=\{x_{2n}\}$ where $n\in N$. Then A and B are disjoint closed subsets of X. By

Urysohn Lemma, there exists $f \in C^*$ such that $f[A]=0$, $f[B]=1$. Then $S \times S$ can not be eventually in $U\left(f, \frac{1}{2}\right)$, so S is not a Cauchy sequence.

Case (2). If S has a cluster point x, then there exists an open set U containing x such that S is frequently in U and $X-U$. Since X is Hausdorff the point x is closed, and by normality, there exists a closed neighborhood V of x such that $V \subset U$. Again by Urysohn Lemma, there exists $f \in C^*$ such that $f[V]=0$, $f[X-U]=1$. Observing that S must be frequently in both V and $X-U$, we conclude that $S \times S$ can not be eventually in $U\left(f, \frac{1}{2}\right)$. Thus S can not bea Cauchy sequence.

We have proved that $(X, \mathscr{U})$ is sequentially complete. Finally, if $(X, \mathscr{U})$ is complete then by [4, Theorem 6-32] X must be compact since $\mathscr{U}$ is precompact. Hence every noncompact normal Hausdorff space admits an incomplete uniform structure in which every Cauchy sequence converges.

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