ON A SUBGROUP OF THE MODULAR GROUP

BY

JAIHAN YOON

1. Introduction. The set of all matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $ad-bc=1$, a, b, c and d are integers, is a group under the matrix multiplication, which is called *the modular group* and denoted by Γ.

It is the purpose of this note to show that the index of the subgroup Γ_{6n} (§ 3) of the group Γ is $6n$ and to find the fundamantal region of Γ_{6n} (§ 4).

2. Some Lemmas

LEMMA 1. *Let* Γ_6 *be the set of all matrices* $\begin{pmatrix} 1+2m_1 & 2m_2 \\ 2m_3 & 1+2m_4 \end{pmatrix}$. *Then,* Γ_6 *is a normal subgroup of* Γ *and its index in* Γ *is* 6.

Proof. A simple calculation shows that Γ_6 is a normal subgroup of Γ. In the course of the calculation we obtain the following equalities which will be used later.

$$\begin{pmatrix} 1+2m_1 & 2m_2 \\ 2m_3 & 1+2m_4 \end{pmatrix}\begin{pmatrix} 1+2p_1 & 2p_2 \\ 2p_3 & 1+2p_4 \end{pmatrix}=\begin{pmatrix} 1+M_1 & 2M_2 \\ 2M_3 & 1+2M_4 \end{pmatrix}, \tag{2.1}$$

where

$$\begin{aligned} M_1&=m_1+p_1+2m_1p_1+2m_2p_3, \\ M_2&=m_2+p_2+2m_1p_2+2m_2p_4, \\ M_3&=m_3+p_3+2m_3p_1+2m_4p_3, \end{aligned} \tag{2.2}$$

and $M_4=m_4+p_4+2m_4p_4+2m_3p_2$.

To prove that the index of Γ_6 in Γ is 6, let A_i $(i=1, 2, \cdots\cdots, 5)$ be the subsets of Γ defined as follows:

$$\begin{aligned} &A_1=\left\{\begin{pmatrix} 1+2m_1 & 2m_2 \\ 1+2m_3 & 1+2m_4 \end{pmatrix}\right\}, \\ &A_2=\left\{\begin{pmatrix} 1+2m_1 & 1+2m_2 \\ 2m_3 & 1+2m_4 \end{pmatrix}\right\}, \quad A_3=\left\{\begin{pmatrix} 2m_1 & 1+2m_2 \\ 1+2m_3 & 2m_4 \end{pmatrix}\right\}, \\ &A_4=\left\{\begin{pmatrix} 2m_1 & 1+2m_2 \\ 1+2m_3 & 1+2m_4 \end{pmatrix}\right\}, \quad A_5=\left\{\begin{pmatrix} 1+2m_1 & 1+2m_2 \\ 1+2m_3 & 2m_4 \end{pmatrix}\right\}. \end{aligned} \tag{2.3}$$

Then, Γ is the disjoint union of Γ_6 and A_i; that is

$$\Gamma=\Gamma_6\cup A_1\cup A_2\cup\cdots\cdots\cup A_5 \tag{2.4}$$

and

$$\begin{aligned} &A_1=\begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}\Gamma_6, \quad A_2=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\Gamma_6 \\ &A_3=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Gamma_6, \quad A_4=\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}\Gamma_6, \\ &A_5=\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}\Gamma_6 \end{aligned} \tag{2.5}$$

Thus Lemma 1 is proved.

LEMMA 2. *Let* $Z=\left\{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\right\}$ *and* $Q=\left\{\begin{pmatrix}1+m_1 & 2m_2\\2m_3 & 1+4m_4\end{pmatrix}\right\}$.

Then, Γ_6/Z *is isomorphic to the group* Q, *which is the the free group generated by*

$$S=\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix} \text{ and } T=\begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}.$$

Proof. That Q is a free group generated by S and T is shown in [3]. Accordingly, we need only to prove that Γ_6/Z is isomnrphic to Q.

The element L of Γ_6 is of the form

$$\begin{pmatrix}1+2m_1 & 2m_2\\2m_3 & 1+2m_4\end{pmatrix},$$

where $m_1+m_4=2(m_2m_3-m_1m_4)$, and hence m_1, m_2 are odd or even simultaneously. Making use of this fact, we define below a map $f\colon \Gamma_6\to Q$ which is actually a homomorphism.

If m_1 and m_4 are even, we have $2m_1=4m_1'$, $2m_4=4m_4'$ for some integers m_1' and m_4'. In this case, the image of L and $-L$ under f is defined by

$$f(L)=f(-L)=\begin{pmatrix}1+4m_1' & 2m_2\\2m_3 & 1+4m_4'\end{pmatrix}.$$

In case of m_1 and m_2 are odd, we define

$$f(L)=f(-L)=\begin{pmatrix}1+4(1-m_1') & 2m_2\\2m_3 & 1+4(1-m_4')\end{pmatrix},$$

where m_1' and m_4' subject to the equality

$$-(1+2m_i)=1+4(1-m_i')\ \ (i=1,4).$$

It is clear that f is a well-defined map of Γ_6 onto Q.

To show that f preserves the multiplication, let

$$L_1=\begin{pmatrix}1+2m_1 & 2m_2\\2m_3 & 1+2m_4\end{pmatrix}, \quad \text{and } L_2=\begin{pmatrix}1+2p_1 & 2p_2\\2p_3 & 1+2p_4\end{pmatrix}.$$

Cases should be divided according as (i) both m_1 and p_1 are even, (ii) m_1 is even but p_1 is odd (iii) m_1 is odd but p_1 is even and (iv) both m_1 and p_1 are odd.

Assume (i). Then, by (2,1), (2,2) and that m_i, p_i $(i=1,4)$ are even or odd simultaneously, we have

$$L_1L_2=\begin{pmatrix}1+2M_1 & 2M_2\\2M_3 & 1+2M_4\end{pmatrix}$$

where

$$M_1=m_1+p_1+2m_1p_1+2m_1p_3=2M_1',$$
$$M_2=m_2+p_2+2m_1p_2+2m_2p_4,$$
$$M_3=m_3+p_3+2m_3p_1+2m_4p_3$$

and
$$M_4=m_4+p_4+2m_4p_4+2m_3p_2=2M_4'$$

for some integers M_1' and M_4'.

Hence, by the definition of f,

$$f(L_1L_2)=\begin{pmatrix}1+4M_1' & 2M_2\\ 2M_3 & 1+4M_4'\end{pmatrix}.$$

On the other hand, a similar calculation gives

$$f(L_1)\cdot f(L_2)=\begin{pmatrix}1+4M_1' & 2M_2\\ 2M_3 & 1+4M_4'\end{pmatrix},$$

showing $$f(L_1L_2)=f(L_1)\cdot f(L_2).$$

Proof for the remaining case are omitted because they parallels that of (i).

Thus f is a homomorphism of Γ_6 onto Q, and it only remains to show that the kernel of f is Z. This, however, is immediate from the very definition of f.

3. The Main Theorems.

DEFINITION. The group generated by the elements of Γ_6 of the form

$$S_{2n}=\begin{pmatrix}1 & 2n\\ 0 & 1\end{pmatrix},\quad U=\begin{pmatrix}1-4n & 2n\\ -2 & 1\end{pmatrix}$$

and

$$U_r=\begin{pmatrix}4r+1 & -8r^2\\ 2 & -(4r-1)\end{pmatrix}\ (r=1,2,\cdots\cdots,n-1)$$

will be denoted Γ_{6n}.

THEORFM 1. *The index of Γ_{6n} in Γ_6 is n.*

Proof. The matrices given in the above definition can be written as follows:

$$S_{2n}=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}^n=S^n,\quad U=\begin{pmatrix}1 & 2n\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}=S^nT^{-1},$$

$$U_r=\begin{pmatrix}1 & 2r\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -2 & 1\end{pmatrix}\begin{pmatrix}1 & -2r\\ 0 & 1\end{pmatrix}=S^rTS^{-r}.$$

Hence the group Γ_{6n} is generated by S^n, U and S^rTS^{-r} $(r=1,2,\cdots\cdots,n-1)$.

Let X be any element of Γ_6. Since Γ_6/Z is the free group generated by S and T, X can be expressed as

$$X=S^{m_1}T^{n_1}S^{m_2}T^{n_2}\cdots\cdots S^{m_r}T^{n_r}$$

where the integers m_1, n_1, $\cdots\cdots$, n_r are all not zero possibly except for m_1 or n_r.

Assume all $m_1, n_1, \cdots\cdots, n_r$ are positive.

If $m_1\geq n$, we have $m_1=nr_1+q_1$, $0\leq q_1<n$ and

$$\begin{aligned}S^{m_1}T^{n_1}S^{m_2}&=S^{nr_1+q_1}T^{n_1}S^{-q_1}S^{m_2+q_1}\\ &=(S_{2n})^{r_1}(S^{q_1}T^{n_1}S^{-q_1})S^{m_2+q_1}\\ &=(S_{2n})^{r_1}(U_{q_1})^{n_1}S^{m_2+q_1}.\end{aligned}$$

If $0<m_1<n$, $S^{m_1}T^{n_1}S^{m_2}$ can be written as follows:

$$\begin{aligned}S^{m_1}T^{n_1}S^{m_2}&=S^{m_1}T^{n_1}S^{-m_1}S^{m_2+m_1}\\ &=(U_{m_1})^{n_1}S^{m_2+m_1}\end{aligned}$$

Now, X reduces to

$$X=(S_{2n})^{r_1}(U_{q_1})^{n_1}S^{m_2+q_2}T^{n_2}S^{m_3}\cdots\cdots$$

or $$X=(U_{m_1})^{n_1}S^{m_2+m_1}T^{n_2}S^{m_3}\cdots\cdots$$

according as $m_1\geq n$ or $m_1<n$ respectively.

A successive application of such a method to X_1 it reduces to

$$X=Y\cdot S^{q_r},\quad Y\epsilon\Gamma_{6n},\quad 0\leq q_r<n$$

That is, $X\epsilon\Gamma_{6n}S^i$ for some i $(0\leq i\leq n-1)$.

A slight modification and the equality $T^{n_1}S^{m_2}=(U^{-1})^{n_1}S^{nm_1+m_2}$ also shows that, for any integers $m_1, n_1, \cdots\cdots, n_r$, the last statement remains true.

Thus we have proved that the index of Γ_{6n} in Γ_6 is n.

An immediate consquence of Theorem 1 is that the index of Γ_{6n} in Γ is $6n$, and

(3.1) $$\Gamma=\Gamma_{6n}\cup\Gamma_{6n}S\cup\Gamma_{6n}S_2\cup\cdots\cdots\cup\Gamma_{6n}S^{n-1}.$$

THEOREM 2. *Matrices $T, U_1, U_2, \cdots\cdots, U_{n-1}$, and S_{2n} generate a free group.*

Proof. If the group is not free, there is a non-trivial word G such that

$$G=T^{m_1^r}U_1^{m_2^r}\cdots\cdots S_{2n}^{m_{n+1}^1}\cdots\cdots T^{m_1^1}U^{m_2^1}\cdots\cdots S_{2n}^{m_{n+1}^1}=E.$$

The word G can be written as

$$G=T^{m_1^r}(ST^{m_2^r}S^{-1})(S^2T^{m_3^r}S^{-2})\cdots\cdots(S^{n-1}T^{m_r^r}S^{-n+1})$$

$$S^{n\cdot m_n^r+1}\cdots\cdots T^{m_1^1}(ST^{m_2^1}S^{-1})(S^2T^{m_3^1}S^{-2})$$

$$\cdots\cdots(S^{n-1}T^{m_n^1}S^{-n+1})S^{n+m_{n+1}^1}$$

and it reduces to

$$G=T^{m_s}S^{m_s}\cdots\cdots T^{m_1}S^{m_1}.$$

The non-triviality leads to a contradiction as in the proof of [2, Theorem 2].

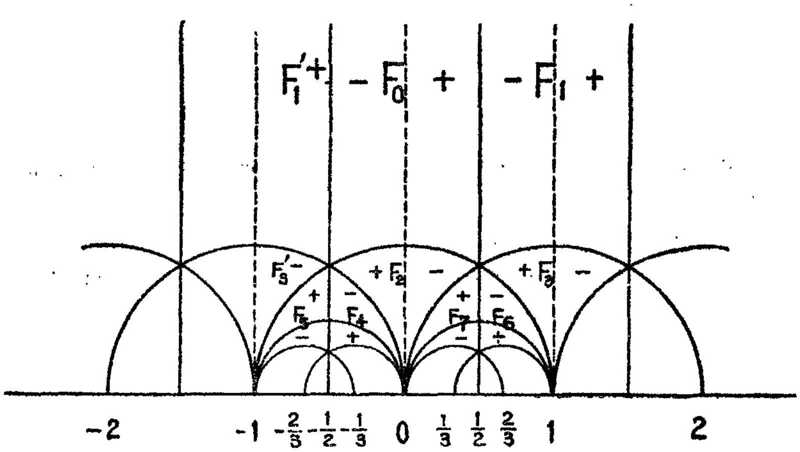
4. The Fundamental Region of Γ_{6n}.

For the notation adopted in this section, the reader is refered to [2].

It is well known that the fundamental region of Γ is F_0 (fig. 1), and, in the light of (2.4) and (2.5), we can state that the fundamental region of Γ_6 is the union

$$F_0\cup F_1\cup\cdots\cdots\cup F_5 \qquad \text{(fig. 1)}$$

Fig. 1

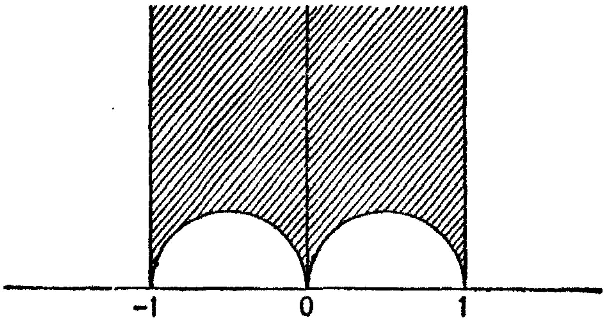


(i) It is clear that the map $\begin{pmatrix}1 & -2\\0 & 1\end{pmatrix}\varepsilon\,\Gamma_6$ sends F_1^+ and F_8^- onto $F'_1{}^+$, $F'_8{}^-$ respectively.

(ii) Clearly $\begin{pmatrix}1 & 0\\2 & 1\end{pmatrix}\epsilon\,\Gamma_6$ maps F_5^-, F_4^+ onto F_6^-, F_7^+ respectively.

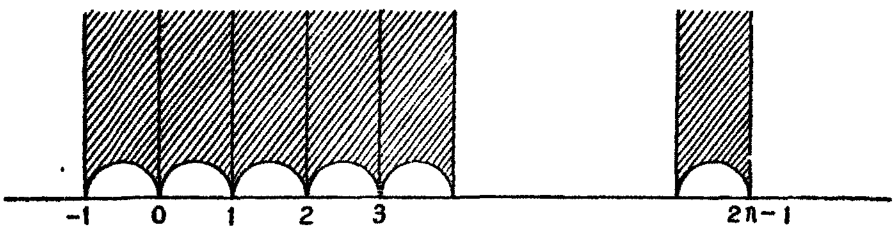
(iii) In view of (i) and (ii), the fundamental region F' of Γ_6 may be considered as the one indicated in Fig. 2.

Fig. 2



(iv) By (3.1) and (iii), the fundamental region of Γ_{6n} may be considered as the one indicated in Fig. 3.

Fig. 3



References

1. B.Chang, S.A. Jennings and R.Ree, *A certain pairs of matrices which generate free groups*, Canadian Journal of Mathematics, 10(1958), 279-284

2. W. Magnus, *Discrete Group*, Lecture Note, New York University, 1952.

3. I. N, Sanov, *A property of a representation of a free group*, Doklady Akad. Nauk(N.S.), 57 (1947), 657-659.