ON COMPLETELY 0-SIMPLE SEMIGROUPS WITH THEIR GRAPHS*

BY

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Let S be a semigroup with 0. Let $a \in S \setminus 0$. Denote by $V(a)$ the set of all inverses of a in S, that is, $V(a)=\{x \in S: axa=a \text{ and } xax=x\}$. A semigroup S with 0 is said to be ***homogeneous n regular*** if the cardinal number of the set $V(a)$ is n for every non-zero element a in S, where n is a fixed positive integer.

Let $P=(p_{ij})$ be any m by m matrix over a group G^0 with 0, and consider any m distinct points $A_1, A_2, \cdots, A_m$ in the plane, which we call vertices. For every non-zero entry $p_{ij} \neq 0$ of the matrix P, we connect the vertex A_i to the vertex A_j by means of a path A_iA_j, which we shall call an ***edge*** directed from A_i to A_j. In this way, with every m by m matrix P can be associated a finite directed graph $G(P)$ of the matrix P.

Let $S=M^0(G; m, m; P)$ be a Rees matrix semigroup over a group G^0 with 0 with a sandwich matrix P. Then the graph $G(P)$ of the sandwich matrix P is called the ***graph*** of the semigroup S.

Let A be a vertex of a graph $G(P)$. The ***local degree*** at a vertex A is the number of edges having A as one end point. In a directed graph there are two types of edges at each vertex A. The outgoing edges from A and the incoming edges to A. Correspondingly we have two local degrees: the number $\phi(A)$ of outgoing edges and the number $\phi^*(A)$ of incoming edges. A directed graph shall be called ***regular of degree n*** when all local degrees $\phi(A)$, $\phi^*(A)$ have the same value $\phi(A)=n=\phi^*(A)$, for every vertex A [4, p. 11].

The purpose of this paper is to prove the following theorem [3].

THEOREM. *A Rees matrix semigroup $S=M^0(G; m, m; P)$ is homogeneous n^2 regular if the directed graph $G(P)$ of the semigroup S is regular of degree n, where m and n are positive integers with $n \leq m$.*

We need the following lemma.

LEMMA. *Let $a=(g)_{ij}$ be a regular element of a Rees matrix semigroup $S=M^0(G;I,J;P)$. Then $|V(a)|=$(number of non-zero entries of the jth row of P)(number of non-zero entries of the ith column of P).*

Proof. Let P_{jx} and P_{xi} be sets of all non-zero entries of the jth row and ith column of

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the sandwich matrix P, respectively. If $a=(g)_{ij}\neq 0$ and if $p_{hi}\in P_{xi}$, $p_{jk}\in P_{jx}$, then $(g)_{ij}\circ(p_{jk}^{-1}g^{-1}p_{hi}^{-1})_{ij}\circ(g)_{ij}=(gp_{jk}p_{jk}^{-1}g^{-1}p_{hi}^{-1}p_{hi}g)_{ij}=(g)_{ij}$, $(p_{jk}^{-1}g^{-1}p_{hi}^{-1})_{kh}\circ(g)_{ij}\circ(p_{jk}^{-1}g^{-1}p_{hi}^{-1})_{kh}=(p_{jk}^{-1}g^{-1}p_{hi}^{-1}p_{hi}gp_{jk}p_{jk}^{-1}g^{-1}p_{hi}^{-1})_{kh}=(p_{jk}^{-1}g^{-1}p_{hi}^{-1})_{kh}$. Hence $(p_{jk}^{-1}g^{-1}p_{hi}^{-1})_{kh}\in V(a)$ and $|P_{jx}|\,|P_{xi}|\leqq|V(a)|$. Conversely, let $b\in V(a)$ and let $b=(g')_{lm}$, for $g'\in G$, $l\in I$ and $m\in J$. From $aba=a\neq 0$ and $(g)_{ij}\circ(g')_{lm}\circ(g)_{ij}=(g)_{ij}$, it follows that $p_{jl}\in P_{jx}$ and $p_{mi}\in P_{xi}$. Now we can see that $|V(a)|=|P_{jx}|\,|P_{xj}|$. This completes the proof of Lemma.

Proof of Theorem. Let $\{A_i: i=1,2,\cdots,m\}$ be the set of vertices of the directed graph $G(P)$ of the semigroup $S=M^0(G;m,m;P)$. If the graph $G(P)$ is regular of degree n, then $\phi(A_i)=\phi^*(A_i)$ for every vertex A_i $(i=1,2,\cdots,n)$, and hence $|P_{jx}|=n=|P_{xi}|$ by the definition of the graph of a square matrix. If $a=(g)_{ij}\neq 0$, then, applying Lemma, $|V(a)|=n^2=|P_{xi}|\,|P_{jx}|$. Since $a\neq 0$ is arbitrary in S, Theorem is proved.

References

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