大韓數學會誌 數學

◇ 論　　文 ◇

SINGULAR INTEGRAL EQUATIONS WITH CAUCHY'S PRINCIPAL VALUE OF AN INTEGRAL AND HILBERT'S TRANSFORMATIONS

BY
HOKEE MINN

1. Introduction. The Fredholm integral equation of the second kind for a function $\phi(x)$ is an equation of the type

$$\phi(x)-\lambda\int_a^b K(x, y)\phi(y)dy=f(x) \qquad (a\leq x\leq b).$$

When the kernel is of type

$$K(x, y)=\frac{H(x, y)}{|y-x|^{\alpha}} \qquad (0<\alpha<1, \ H \text{ bounded})$$

it is well known that it can be transformed into a Fredholm type with a bounded kernel. However, in the important case $\alpha=1$ (in which the integral of the equation must be considered as a Cauchy principal integral) the integral equation differs radically from a Fredholm type with a bounded kernel. That is the kernel (with $\alpha=1$) becomes infinite at an interior point $x=x_0$ of the interval of integration (a, b). Therefore we call this type the singular integral equations with Cauchy's principal value of an integral.

The purpose of this paper is to consider this singular type and solve some general case with a use of theory of analytic function, in particular, with the finite Hilbert transformations.

The theory of Hilbert transform

$$f(x)\equiv\frac{1}{\pi}P\int_{-\infty}^{\infty}\frac{\phi(y)}{y-x}dy=H_x[\phi(y)]$$

where P denotes the Cauchy's principal value, is discussed by Titchmarsh in his book on Fourier integral.

The finite Hilbert transform

$$f(x)\equiv\frac{1}{\pi}P\int_{-1}^{1}\frac{\phi(y)}{y-x}dy=\mathscr{T}_x[\phi(y)]$$

is less well-known, but it is studied by Tricomi, we quote some of his results which we use later. Consider

$$F(z)=\frac{1}{\pi}\int_{-1}^{1}\frac{\phi(t)}{t-z}dt$$

where z is generally a complex number lying outside the segment $(1,1)$ on the real axis. This transform changes a real function of the class L_p $(p>1)$, where L_p is Lebesgue class of p, into a single-valued analytic function which is regular in the whole z-plane cut along the segment $(-1, 1)$ of the real axis, vanishes at infinity, and satisfies the condition

$$\int_{-\infty}^{\infty} |\phi(x+iy)|^p dx < K \qquad (p>1)$$

for all values of y, $K=$const.>0. Then it is shown that

$$\frac{1}{2}[\phi(x+i\epsilon)+\phi(x-i\epsilon)]=\frac{1}{2i}[F(x+i\epsilon)-F(x-i\epsilon)].$$

In other words, since $F(x+i\epsilon)$ and $F(x-i\epsilon)$ are conjugate complex numbers for ϕ real, we can state that

$$Im\ F(x+i\epsilon)=\frac{1}{2}[\phi(x+i\epsilon)+\phi(x-i\epsilon)] \qquad (-1<x<1)$$
$$=0 \qquad (x<-1 \text{ or } x>1),$$

that is we have discontinuities across the real axis $(-1<x<1)$.

From the above relation

$$Re\ F(x+i\epsilon)=H_x[Im\ F(y+i\epsilon)]=\mathcal{T}_x[\phi^*(y)]=\mathcal{T}_x[\phi(y)]$$

hence

$$F(x+i\epsilon)=\mathcal{T}_x[\phi(y)]+i\phi^*(x) \qquad (-1<x<1),$$

where

$$\phi^*(x)=\frac{1}{2}[\phi(x+i\epsilon)+\phi(x-i\epsilon)],$$

H_x, $\mathcal{T}_x$ are the operator of Hilbert transform, finite Hilbert transform respectively, and we use the famous Hilbert relations between the real and imaginary part of an analytic function. From this there follows the formula which for ϕ continuous can be written

$$F(x+i\epsilon)=\frac{1}{\pi}P\int_{-1}^{1}\frac{\phi(y)}{y-x}dy \pm i\phi(x)$$

In the operator form it is the well-known formula

$$\frac{1}{x\pm i\epsilon}=P\frac{1}{x}\mp i\pi\,\delta(x).$$

2. Singular Integral Equation. Given a singular integral equation of the second kind with the singular kernel

$$K(x,y)=\frac{H(x,y)}{y-x} \qquad (1)$$

we can expand this in Taylor series about x as follows

$$H(x,y)=H(x,x)+(y-x)H_y'(x,x)+\frac{1}{2!}(y-x)^2H_{yy}''(x,x)+\cdots\cdots\cdots$$

so that

$$K(x,y)=\frac{H(x,x)}{y-x}+K^*(x,y) \qquad (2)$$

where $K^*(x,y)$ is bounded.

Consequently the main problem in studying integral equations with kernels of type (2) is solving the standard equation

$$a(x)\phi(x)-\lambda P\int_{-1}^{1}\frac{\phi(y)}{y-x}dy=f(x) \tag{3}$$

where

$$a(x)=\frac{1}{H(x,x)} \tag{4}$$

The following method is similar to the solving Faltung type of equations by means of Laplace transform. We use the finite Hilbert transform here.

If we put

$$F(z)=\frac{1}{\pi}\int_{-1}^{1}\frac{\phi(y)}{y-z}dy \tag{5}$$

then by the result of the previous section, we have

$$\left.\begin{aligned}&F(x+i\epsilon)-F(x-i\epsilon)=2i\phi(x)\\&F(x+i\epsilon)+F(x-i\epsilon)=\frac{2}{\pi}P\int_{-1}^{1}\frac{\phi(y)}{y-x}dy\end{aligned}\right\} \tag{6}$$

where we assumed that $\phi(x)$ is continuous in the open interval $(-1,1)$. Consequently equation (3) assumes the algebraic form

$$[a(x)-\lambda\pi i]F(x+i\epsilon)-[a(x)+\lambda\pi i]F(x-i\epsilon)=2if(x). \tag{7}$$

This equation can be simplified by setting

$$F(z)=e^{T(z)}U(z) \tag{8}$$

provided that the function $T(z)$ satisfies the condition

$$[a(x)-\lambda\pi i]e^{T(x+i\epsilon)}=[a(x)+\lambda\pi i]e^{T(x-i\epsilon)} \tag{9}$$

We obtain

$$U(x+i\epsilon)-U(x-i\epsilon)=\frac{2if(x)}{a(x)-\lambda\pi i}e^{-T(x+i\epsilon)}=\frac{2if(x)}{a(x)+\lambda\pi i}e^{-T(x-i\epsilon)}$$

from which, if we consider the geometric mean of the two expressions for the difference on the left(which are equal), it follows that

$$U(x+i\epsilon)-U(x-i\epsilon)=\frac{2if(x)}{[a^2(x)+\lambda^2\pi^2]^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}[T(x+i\epsilon)+T(x-i\epsilon)]\right\}. \tag{10}$$

How, in order to determine the function $T(z)$, we observe that from (9) we have

$$T(x+i\epsilon)-T(x-i\epsilon)=\log\frac{a(x)+\lambda\pi i}{a(x)-\lambda\pi i}=2i\tan^{-1}\frac{\lambda\pi}{a(x)}. \tag{11}$$

Consequently, we can put

$$T(z)=\frac{1}{\pi}\int_{-1}^{1}\frac{\theta(t)}{t-z}dt \tag{12}$$

with

$$\theta(x)=\tan^{-1}\frac{\lambda\pi}{a(x)}.$$

It follows from this that

$$\frac{1}{2i}[T(x+i\epsilon)-T(x-i\epsilon)]=\theta(x)=\tan^{-1}\frac{\lambda\pi}{a(x)}$$

in accordance with (11).

On the other hand, we have

$$\frac{1}{2}[T(x+i\epsilon)+T(x-i\epsilon)]=\frac{1}{\pi}P\int_{-1}^{1}\frac{\theta(t)}{t-x}dt,$$

hence, equation (10) becomes now

$$U(x+i\epsilon)-U(x-i\epsilon)=\frac{2i}{[a^2(x)+\lambda^2\pi^2]^{\frac{1}{2}}}e^{-\tau(x)}f(x),$$

where

$$\tau(x)=\frac{1}{\pi}P\int_{-1}^{1}\frac{\theta(t)}{t-x}dt \qquad (14)$$

and can be satisfied by the function

$$U(z)=\frac{1}{\pi}\int_{-1}^{1}\frac{e^{-\tau(t)}f(t)}{[a^2(x)+\lambda^2\pi]^{\frac{1}{2}}}\cdot\frac{t-z}{dt}.$$

Finally, we determine ϕ by using the first equation (6) which, in view of (12) and (15), gives us

$$2i\phi(x)=e^{T(x+i\epsilon)}\ U(x+i\epsilon)-e^{T(x-i\epsilon)}\ U(x-i\epsilon)$$

$$=e^{\tau(x)+i\theta(x)}\Big[\frac{1}{\pi}P\int_{-1}^{1}\frac{e^{-\tau(y)}f(y)}{[a^2(y)+\lambda^2\pi^2]^{\frac{1}{2}}}\quad\frac{dy}{y-x}+i\frac{e^{-\tau(x)}f(x)}{[a^2(x)+\lambda^2\pi^2]^{\frac{1}{2}}}\Big]$$

$$-e^{\tau(x)-i\theta(x)}\Big[\frac{1}{\pi}P\int_{-1}^{1}\frac{e^{-\tau(y)}(y)}{[a^2(y)+\lambda^2\pi^2]^{\frac{1}{2}}}\quad\frac{dy}{y-x}-i\frac{e^{-\tau(x)}f(x)}{[a^2(x)+\lambda^2\pi^2]^{\frac{1}{2}}}\Big]$$

after making some simplications, we get

$$\phi(x)=\frac{a(x)\ f(x)}{a^2(x)+\lambda^2\pi^2}+\frac{\lambda e^{T(x)}}{[a^2(x)+\lambda^2\pi^2]}P\int_{-1}^{1}\frac{e^{-\tau(y)}f(y)}{[a^2(y)+\lambda^2\pi^2]}\quad\frac{dy}{y-x} \qquad (16)$$

where, according to (14) and (13),

$$\tau(x)=\mathscr{T}_x[\theta(y)],\ \theta(y)=\tan^{-1}\frac{\lambda\pi}{a(y)}$$

3. Conclusions. Main interest of this method is that we get the solution in closed form with relatively simple mathematical labour. The various special case of the above type of singular integral equations arise in the many different branches of applied mathematics, physics and engineering.

For example, S. Chandrasekhar has found important applications for these equations to problem of radiative equilibrium of a steller atmosphere. Recently, also the special case of this type occurs in socalled "dispersion relation" type equation, Omnes' equation, where Hilbert transform are extensively used. Also Russian mathematical school (e. g. Vekua, Muskhelishvili, Mihlin and Kupradze) is interested in this field, and made important contributions for equations with a Cauchy principal integral over a closed curve in the complex plane.

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Seoul National University