GEOMETRICAL REPRESENTATIONS OF POINTS IN THE FIELD OF COMPLEX QUANTITIES

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O. INTRODUCTION

The Purpese of this Article: The main purpose of this article is to contruct the primitive propositions of the algebra of complex quantities which is to be the foundation of algebra. It is intended as a systematic introduction to the algebra of complex quantities with a number of geometrical representations of points in the plane.

We have the algebra of positive integers; the algebra of all integers (positive, negative, zero); the algebra of positive rationals; the algebra of all rationals, the algebra of all real quantities (rational and irrational, positive, negative and zero); and, finally, the algebra which includes all these and others, and is, in many respects simpler than any of them, *the algebra of Complex quantities.*

Plan of the article: Complex quantities may be represented by points in the plane. The position of any point a in the plane is determined by the angle and the distance from the origin.

In I, as a preliminary part for the main discussion (II and III), operations with angles and distances are given and some conditions, like postulates for an abstract science, to include addition and multiplication are given, In §3 of this part several examples are given that satisfy most, but not all, of the conditions of §3.

In II, a number of geometrical facts are observed. II shows how this collections of geometrical facts can be reduced to an abstract science, and serves to illustrate, in this very simple case, all the steps of the resoning which will be used in general case in III.

III shows a complete set of primitive propositions for the algebra of complex quantities with an analysis of the fundamental concepts which occur in modern algebrea; such as the isomorphism of two algebraic systems with respect to these fundamental concepts, the sufficiency, consistency and independence of the propositions of §8.

I. OPERATIONS OF ANGLES AND DISTANCES

§1. The addition of angles

We begin with a preliminary discussion of the very simple and familiar process of the addition of angles.

Definition: If two angles α and β are given, a third angle γ which is the sum of α and β may be derived by the following process:

Starting with a given initial line as the zero angle, perform the rotation indicated by α; then continuing from the terminal line of α, perform a rotation equal in amount and direction to β; the final position thus reached is the terminal line of the required angle γ.

Concernig the addition of angles we may easily verify the following familiar statements:

A_1: If α and β are any two angles, then their sum, $\alpha+\beta$, is an angle uniquely determined by α and β.

A_2: $\alpha+\beta=\beta+\alpha$. (Commutative law for addition)

A_3: If α, β, γ are any three angles, $(\alpha+\beta)+\gamma=\alpha+(\beta+\gamma)$ (Associative law for addition)

A_4: If $\alpha+\beta=\alpha+\beta'$, then $\beta=\beta'$ (Law of cancellation for addition)

If we introduce the notation, $2\alpha=\alpha+\alpha$, $3\alpha=\alpha+\alpha+\alpha,\dots$, $n\alpha=\alpha+\alpha+\dots+\alpha$ to n terms, where n is any positive integer, we have further:

A_5: If $n\alpha=n\beta$, then $\alpha=\beta$.

The angle $n\alpha$ is called the *nth multiple* of the angle α.

A_6: There is one and only one angle x such that $x+x=x$; this angle x is called the *zero* angle, and is denoted by 0

A_7: Every angle α determines uniquely angle α' such that $\alpha+\alpha'=0$. This angle α' is called the *opposite* of α and is denoted by $-\alpha$.

A_8: For every angle α and every positive integer n, there is an angle y, uniquely determined by α and n, such that $ny=\alpha$. This angle is called the *nth submultiple of* α, denoted by α/n.

Now if we agree to take the propositions A_1—A_8 as axioms for the addition of angles, then the other propositions can be deduced as theorems, for exemple;

A_9: If 0 is the zero angle, then for every angle α,

$$\alpha+0=\alpha.$$

Proof: By A_6 $\quad 0+0=0$, hence by A_1,

$$\alpha+(0+0)=\alpha+0$$

and by A_3 $\quad (\alpha+0)+0=\alpha+0$,

whence, by A_2 $\quad 0+(\alpha+0=0+\alpha$.

Therefore by A_4 $\quad \alpha+0=\alpha$,

which was to be proved.

A_{10}: If α and β are any given angles, there is always an angle x uniquely determined by α and β, such that $\alpha=\beta+x$; this angle x is called the *remainder* and is denoted by $\alpha-\beta$.

Proof: By A_6 and A_7, there is an angle $-\beta$ such that

$$\beta+(-\beta)=0.$$

Let $x=\alpha+(-\beta)$, which is known to be an angle, by A_1 Then, by the use of A_2, A_3, A_7, and A_2, and the A_9,

$$\beta+x=\beta+[\alpha+(-\beta)]=\beta+[(-\beta)+\alpha]=[\beta+(-\beta)]+\alpha=0+\alpha=\alpha+0=\alpha;$$

which was to be proved.

That this angle is uniquely determined by α and β follows at once from A_4.

§ 2. The multiplication of distances

Definition: Suppose two distances a and b are given; and then, having chosen a given distance u as *unit distance*, find a distance x by the construction in Fig. 1, in which b is at right angles to u and a, and the oblique lines are parallel.

This distance x is called the *product* of the given distances a and b (with respect to the chosen unit) and denoted by $a\times b$, or simple ab.

From this definition it follows that if $x=a\times b$, the area of the rectangle whose sides are x and u is equal to the area of the rectangle whose sides are a and b.

Proof:

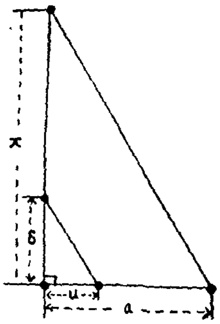

Fig. 1

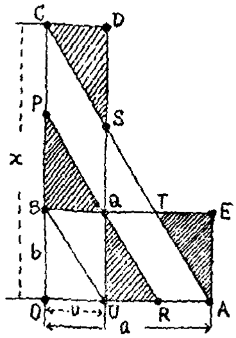

Fig. 2

To show *that the two rectangles, OCDU and OBEA, have equal area, note that the part OBQU* is common to both; further, the lines PQ, QR, CR, and TA are all equal to BU (being portions of parallels intercepted between parallels), so that the triangles BPQ and DSC in one rectangle are equal to the triangles UQR and ETA in the other; and finally, the parallelograms CSQP in one rectangle and QTAR in the other have equal area(having equal bases PQ and QR and equal altitudes).

Concerning the multiplication of distances, as thus defined, we may verify the following statements:

M_1: If a and b are any two distances then their product, $a\times b$, is a distance uniquely determined by a and b (with respect to the chosen unit distance).

M_2: $a\times b=b\times a$. (Commutative law for multiplication)

M_3: If a, b, c are any three distances, then

$(a\times b\times)c=a\times(b\times c)$. (Associative law for multiplication)

Proof: *Let* $a\times b=x$ and $b\times c=y$, and then $x\times c=z$ and $a\times y=z'$, so that we have $(a\times b)\times c=x\times c=z$ and $(a\times b\times c\times)=a\times y=z'$.

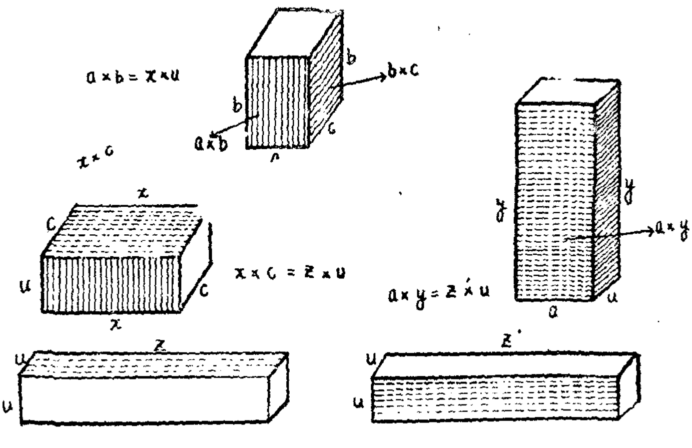

Fig. 4

Then we can see that the parallelopiped whose edges are z, u, u and the parallelopiped whose edges are z, n, u both have the same volume as the parallelopiped whose edges are a, b, c.

$$\therefore\ z=z'$$

M_4: If $a\times b=a\times b'$, then $b=b'$. (Law of cancellation for multiplication).

If we introduce the notation $a^2=a\times a$, $a^3=a\times a\times a$, ..., $a^n=a\times a\times ...\times a$ to n factors, where n is any positive integer, we have further:

M_5: If $a^n=b^n$, then $a=b$. The distance a^n is called the *nth power* of the distance a

M_6: There is one and only one distance x such that $x\times x=x$.

This distance x is the *unit distance*, and is denoted by 1.

M_7: Every distance a determines uniquely a distance a' such that $a\times a'=1$.

This distance a' is called the *reciprocal of a* and is denoted by a^{-1} or $1/a$.

M_8: For every distance a and every positive integer n, there is a distance y, uniquely determined by a and n, such that $y^n=a$.

This distance y is called the *n th root of a*, denoted by $a^1/^n$ or $\sqrt[n]{a}$.

M_9: If 1 is the unit distance,them for every distance a, $a\times 1=a$.

M_{10}: If a and b are any given distances, there is always a distance x, uniquely determined by a and b, such that $a=b\times x$; this distance x is called the *quotient* and is denoted by a/b.

The distance a/b is the same as the distance $a\times((b^{-1})$. Hence, to devide by the distance b means to multiply by the reciprocal of b.

§ 3. Conditions for Addition or Multiplication.

	Angle +		Distance ×
A_1	closed	M_1	closed $a\times b$
A_2	commutative for Add.	M_2	commutative for Mult.
A_3	Associative for Add.	M_3	Associative for Mult.
A_4	Law of Cancellation	M_4	Law of Cancellation
A_5	sum (n th multiple)	M_5	product (nth power)
A_6	zero 0	M_6	unit 1
A_7	opposite −	M_7	reciprocal a^{-1}
A_8	subtraction	M_8	division a/b

This duality between the two sets of propositions will of course extend through all the propositions that are deducible from them by the method of formal logic. This duality at once suggests the possibility of developing a general theory. To do this I proceed as follows:

Consider a general class of things or elements denoted by A,B,C, etc., without specifying whether these things are to be angles (α, β, γ,...) or distances (a, b, c,...), and a general rule of combination denoted by · whithout specifying whether this rule of combination is to be addition or multiplication and impose upon these symbols the following conditions:

C_1: If A and B are elements of the class, then A · B is an element of the class, uniquely determined by A and B.

C_2: A · B=B · A (Commutative law)

C_3: $(A \circ B) \circ C = A \circ (B \circ C)$ (Associative law)

C_4: If $A \circ B = A \circ B'$, then $B = B'$. (Law of Cancellation)

C_5: If $A^{[n]} = B^{[n]}$, then $A = B$.

Here $A^{[n]}$ means $A \circ A \ldots \circ A$, to n elements, where n is a positive integer.

C_6: There is an element X such that $X \circ X = X$.

It can be shown from the preceding conditions that there cannot be more than one such element.

For, suppose these were two such elements, X and Y, such that $X \circ X = X$ and $Y \circ Y = Y$ then

by C_1, $(X \circ X) \circ Y = X \circ (Y \circ Y)$,

whence, by C_3 $X \circ (X \circ Y) = X \circ (Y \circ Y)$

whence, by C_4 $X \circ Y = Y \circ Y$.

Therefore, by C_2 $Y \circ X = Y \circ Y$,

by C_4 $X = Y$.

C_7: If X is the unique element such that $X \circ X = X$, then for every element A there is an element A' such that $A \circ A' = X$. (It follows from C_4 that this element A' is uniquely determined by A)

C_8: For every element A and every positive interger n, there is an element Y such that $Y^n = A$, where Y^n means $Y \circ Y \circ Y \circ \ldots \circ Y$ to n elements.

(It follows from C_5 that this element Y is uniquely determined by A and n.)

In fact, the system of angles under addition and the system of distances under multiplication are only two examples out of many which satisfy all these eight conditions, so that we may well assured that the conditions are consistent. These eight conditions, C_1-C_8, may therefore be taken as the fundamental conditions of an abstract science, which will exhibit the logical structure of a large class of systems.

Further, if any system, consisting of a class of elements A,B,C,... and a rule of combination $\circ$, is laid before us, we have only to assure ourselves that this system satisfies the eight conditions of our abstract science, in order to be convinced that this system will also satisfy all the derived theorems, which form the body of the science.

Concerning the set of conditions C_1-C_8 of this section, it will be instructive to give here a few examples of systems which do not satisfy all of these conditions. I shall exhibit eight systems, each of which satisfies all but one of the eight conditions.

Example for C_1: Class=all angles between $-10°$ and $+10°$ $A \circ B = A + B$.

Note. This system fails to satisfy C_1, since $7° \circ 8° = 15°$, for instance, is not in the class. All the other conditions are satisfied.

Example for C_2: Class=positive integers.

$A \circ B = B$. For exemple, $7 \circ 8 = 8$, $15 \circ 3 = 3$, etc.

Note. This system clearly fails to satisfy the commutative law, but all of the other conditions are satisfied. Thus, in C_6, any element X will have required property $X \circ X = X$; since this element X is not uniquely determined. C_7 has nothing further to demand; this is, therefore, satisfied vacuously. We can show that C_8 is satisfied, if we take $Y = A$.

Example for C_3: Class=all angles. $A \circ B = (A+B)/3$.

Note. Here the associative law is not satisfied, since, for example, $(3° \circ 6°) \circ 12° = 3° \circ 12° = 5°$ while $3° \circ (6° \circ 12°) = 3° \circ 6° = 3°$. All the other conditions are satisfied. Thus, in C_6, take X=the zero angle; in C_7, take $A'=-A$; in C_8, notice first that $A^{[2]}=\frac{2}{3}A$ $A^{[3]}=\left(\frac{1}{3}+\frac{2}{3^2}\right)A$,

$A^{[4]}=\left(\frac{1}{3}+\frac{1}{3^2}+\frac{2}{3^3}\right)A$, so that in general, by the formula for the sum of a geometric series, $A^{[n]}=\frac{3^{n-1}+1}{2\times 3^{n-1}}A$

hence, if we take $Y=\frac{2\times 3^{n-1}}{3^{n-1}+1}A$, then C_8 will be satisfied.

Example for C_4: Class=all angles.

If A is distinct from B, $A \circ B$=the zero angle, but $A \circ A=A$.

Note. This system fails to satisfy the law of cancellation, but satisfies all the other conditions. C_7 is satisfied vacuously, since there is no uniiquely determined element X to which this condition could refer.

Example for C_5: Class=all angles; congruent angles being regarded as equal*. $A \circ B$=that angle in the first revolution which is congruent to A+B.

Note. Here C_5 is not satisfied, since, for example, $60^{\circ[2]}=60^\circ \circ 60^\circ=120^{\circ\circ}$ and also $240^{\circ[2]}=240^\circ \circ 240^\circ=120^\circ$, while 60° and 240° are not equal angles. All the other conditions are satisfied.

Example for C_6: Class=all positive distances.

$A \circ B$=the hypotenuse of a right triangle of which A and B are the legs.

Note. Here C_6 is not satisfied, since the hypotenuse of a right triangle is never equal to a leg. All the other conditions are satisfied, but C_7 vacuously.

Example for C_7: Class=all positive angles and the zero angle.

Rule of conbination, $\circ=+$.

Note. This system clearly does not satisfy C_7, since if $A=10^\circ$, for example, the opposite of A is not in the class. All the other conditions are satisfied.

Example for C_8: Class=all integral numbers.

$\circ=+$, where + means the ordinary "+" of arithmetic.

Note. This system fails on C_8, since, for example, there is no integral number y such that $y+y+y=5$. It clearly satisfies all the other conditions.

The examples just cited show that in this case the conditions are all *independent*: for, if C_1, for example, were a consequence of the other seven conditions, then every system which satisfied the other seven would also satisfy C_1, but this is not the case, as is shown by the example cited. Therefore each one of the eight conditions is shown to be independent of the rest.

We have been dealing with systems consisting of a class, say K, and a rule of combination, $\circ$; and among these systems (K, $\circ$) we have found some that satisfy the conditions, and some that do not.

Two systems (K, $\circ$) and (K, $\circ$) being called *isomorphic* if the elements of the class K can be paired off (put into one-to-one correspondence with) the elements of the class K′ in such a way that whenever A and B in the class K correspond to A′ and B′ in the class K′, then $A \circ B$ in K will correspond to $A' \circ B'$ in K′.

* Congruent angles are those that differ only by a multiple of 360°.

II. REAL AND IMAGINARY POINTS

The position of any point a in the plane is determined when we know:

(1) the *distance* of a from O (the distance OU being taken as the unit of measurement),

(2) the *angle* which the line Oa makes with the axis OU.

Two points are "equal", that is, concident, when and only when their distances are equal and their angles equal or congruent

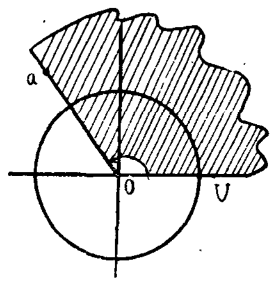

Fig. 5

All the points whose distances equal OU called points on the "unit circle".

§4. Addition of points in the plane.

Definition: If two points a and b are given, the sum of a and b, namely, $x=a+b$, may be derived by the following process:

Starting from O, perform the journey from O to a; then continuing from a, perform a journey equal to length and direction to the journey from O to b; the point finally reached is the required point x.

Concerning the addition of points in the plane, as thus defined, we may easily verify the following statements:

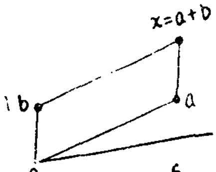

Fig. 6

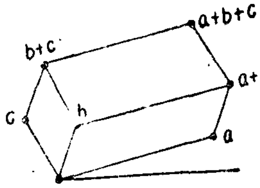

Fig. 7

A_1: If a and b are any two points, then their sum, $a+b$, is a point, uniquely determined by a and b and if a and b are real points, then $a+b$ is also real.

A_2: $a+b=b+a$. (Commutative law for addition)

A_3: $(a+b)+c=a+(b+c)$ (Associative law addition)

These facts will be clear from the accompanying figures.

A_4: If $a+b=a+b'$, then $b=b'$ (Law of cancellation for addition)

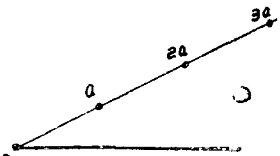

Fig. 8

A_5: If $na=nb$, then $a=b$. Here n is any positive interger, and na means $a+a+\dots a$, to n terms. The point na is called the *nth multiple* of a.

A_6: There is a unique point z such that z+z=z. This poin z is called the *zero* point of the system, and is denoted by 0.

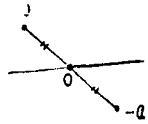

Fig. 9

A_7: Every point a determines uniquely a point a' such that $a+a'=0$.

This point a' is called the *opposite* of a, and is denoted by $-a$.

A_8: If a point a and a positive integer n are given, there is always a point x such that $nx=a$.

This point x is called the nth submultiple of a, and is denoted by a/n.

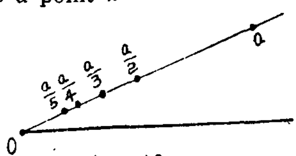

Fig. 10

A_9: If a and b are any two points, there is always a point x such that $a=b+x$.

This point x which is uniquely determined by a and b, is called the remainder and is denoted by $a-b$. (To subtract a point b means to addd the opposite of b.

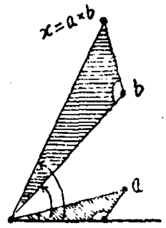

Fig. 11

§ 5. Multiplication of points in the plane.

Definition: If a and b are any two points in the plane, the product of a and b, $x=a\times b$, may be derived by the following process;

Find the angle of x by taking the sum of the angles of a and b. And, find the distance of x, by taking the product of the distances of a and b, as defined in Part I.

M_1: If a and b are any two points then then their product $a\times b$ is a point uniquely determined by a and b.

M_2: $a\times b=b\times a$. (Commutative law for multiplication)

M_3: $(a\times b)\times c=a\times(b\times c)$ (Associative law for multiplication)

M_4: If $a\times b=a\times b'$ and a is not 0 then $b=b'$. (Cancellation law for multiplication)

M_5: $a\times(b+c)=(a\times b)+(a\times c)$ (Distributive law of multiplication with respect to addition)

Proof: To see that this distributive law holds, let each of the points b, c, and $b+c$ be multiplied by a, as in Fig. 12. Place the quadrilateral 0, ab, ac, $a(b+c)$, together with the parallelogram $0,b,c$, $(b+c)$, in a plane perpendicular to the line OU, in the manner shown in Fig. 13, and lay off the distance $0a$ along that line.

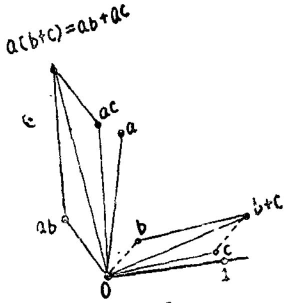

Fig. 12

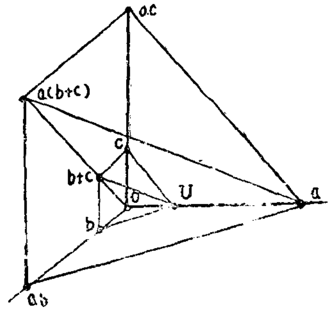

Fig. 13

By the definition of the multiplication of distances, the lines U-c and a-ac are parallel, as are also the lines U$-(b+c)$ and $a-a(b+c)$; therefore the planes $a-ac-a$(b+c) and U$-c-(b+c)$ are parallel, and hence the lines $ac-a$(b+c) and $c-(b+c)$, in which these planes intersect the given plane, are parallel.

Hence $ac-a(b+c)$ is parallel to 0-ab; and similarly, ab-- $a(b+c)$ is parallel to 0--ac. Therefore the quadrilateral in question is a parallelogram, and the point $a(b+c)$ is the sum of the points ab and ac, as required.

M_6: There is a unique point u, distinct from 0, such that $u\times u=u$;

this point u is called the unit point of the system, and is denoted by 1.

If a is any point, $a\times 1=a$.

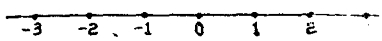

Fig. 14

The successive multiples of the point 1 are denoted as follows:

$1+1=2(1)=2$; $1+1+=3(1)=3$; etc.

M_7: Every point a, provided a is not 0, determines uniquely a point a' such that $a\times a'=1$, where 1 is the unit point.

This point a' is called the reciprocal of a, and is denoted by a^{-1} or $1/a$.

If a is a real point (not 0) then its reciprocal will also be real.

To construct the point $1/a$, we must notice that its angles is the opposite of the angle of a, while its distance is the reciprocal of the distance of a.

If a is a point on the unit circle, then a^{-1} will also be on the unit circle; while if a is inside the circle, a^{-1} will be outside and the nearer a approaches the point 0, the farther off will a^{-1} recede.

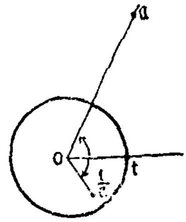

Fig. 15

M_8 : If a and b are any points, and b not 0, then there is always a point x such that

$$a=b\times x.$$

This point x, which is uniquely determined by a and b, is called the quotient and is denoted by a/b.

To construct this point a/b, we have to notice that its angle must be the angle of a minus the angle of b, while its distance must be the distance of a divided by the distance of b.

M_9 : If a is any point, and 0 is the point, them $a\times 0=0$;

and if a product $a\times b=0$, then at least one of the factors a and b must be 0.

M_{10} : The notation a^n, where n is any positive integer, means $a\times a\times a\times a\times\ldots\times a$ to n factors; and the point a^n is called the *nth power of the point a.*

Obviously, from the definition of multiplication, $1^n=1$, and $0^n=0$.

To construct the point a^n, we notice that the angle of a^n is the nth multiple of the angle of a; while the distance of a^n is the nth power of the distance of a.

If the point a lies on the unit circle, then a^n will also lie on this circle; if the point a lies outside the circle, then the series of powers, a, a^2, a^3,... will lie outside the circle, on a spiral curve which recedes farther from it; if the point a lies inside the circle, the series a, a^2, a^3,... will lie inside the circle, on a spiral which again recedes farther and farther from the circle, coiling up around the point 0.

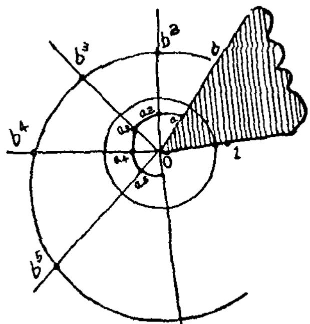

Fig 16

Definition: There are two points x such that $x^2=-1$, where -1 is the opposite of the unit point 1. These two points are called the *imaginary units* of the system, and are denoted by i and $-i$.

Definition: If a is any point, there are always two real points, x and y, such that

$$a=x+iy$$

where i is one of the imaginary units.

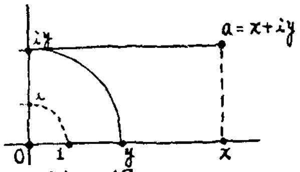

Fig. 17

§ 6. Solution of an nth root of a.

If n is any positive integer, and a is any point not 0, there will be n distinct points x such that $x^n=a$; each of these points is called *an nth root of a.*

Thus, every point a, except 0, has two square roots, three cube roots, four fourth roots, and so on.

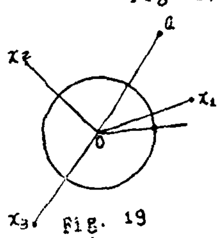

Fig. 19

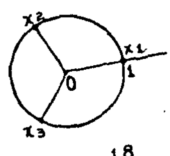

Fig. 18

To construct these points we notice first that if $a=1$, the nth roots of 1 are points on the "*unit circle*," and divide that circle into n equal parts, beginning with the point 1; for, any one of these points, when raised to the nth power according to the rule, will produce the point 1.

In general, the nth roots of any point a will be on a circle whose radius is the nth root of the distance of a, and will divide this circle into n equal parts, beginning with the point whose angle is the nth submultiple of a.

If any one of the nth roots is given, the rest can be obtained from it by multiplying by the nth roots of the point 1.

The notation, $a^{1/n}$, or $\sqrt[n]{a}$ is used to denote that particular nth root of a which has the smallest(positive) angle. $i=\sqrt{-1}$, $-i=-\sqrt{-1}$.

Thus, if a is real, and n is an odd number, one of the nth roots of a will be real, and will be positive or negative according as a is positive or negative. If a is a positive real, and n is even, two of the nth roots of a will be real, one positive and one negative; but if a is a negative real, and n is even, none of the nth roots of a will be real.

§ 7. The Relation of Order among the Real Points.

If the point a precedes the point b as we progress along the axis of reals in the direction OU, then we write $a<b$.

Concerning this relation of serial order among the points on the axis of reals, the following statements are evident.

O_1: If a and b are real points, and a not equal to b, then $a<b$ or $b>a$.

O_2: If $a<b$, then a is not equal to b.

O_3: If $a<b$ and $b<c$, then $a<c$. (Law of transitivity)

O_4: If a, x, and y are real points, and $x<y$, then $a+x<a+y$.

O_5: If $a>0$ and $b>0$. then $a\times b>0$.

O_6: If $a<b$, there are always real points x such that $a<x$ and $x<b$.

Such point x are said to lie between the points a and b.

O_7: (Dedekind's principle)

If M is any (non-empty) subclass of real points, and if all the points of M precede a given point c, then there will be a unique determined point x, called the upper limit of M, having the following properties:

Fig. 20

1) every point in M precedes, or at most equals, x;
2) if x' is any real point such thta $x'<x$, then there is at least one point of M that follows x.

In other words, if a subclass of real points has any "upper bound", it will have a "least upper bound", or "upper limit". Similarly, a subclass of real points that has any lower bound, will have a "greatest lower bound", or "lower limit".

O_8: (Principle of Archimedes)

If a and b are any positive points, and a is less than b, it is always possible to find some multiple of a which is greater than b. i.e. $na>b$.

III. FUNDAMENTAL NOTIONS IN ALGBBRA OF COMPLEX QUANTITIES

The system of points in the plane, mentioned in Part II, is the best known and most easily understood example of the type of algebra called the algebra of complex quantities.

I now proceed to analyze what is logically essential in this system.

The fundamental notions of the system are: the class of points in general, K; the class of real points, R; the operations of addition and multiplication; and the relation of order.

§8. A complete set of Primitive Propositions for the Algebra of Complex Quantities.

Let K : a class of elements, a, b, c,... (i.e. a set of points in general)

R : a class of elements. (i.e. a set of all real points)

$\oplus$: a rule of combination.

$\otimes$: a rule of combination.

$\oslash$: a relation.

I want to show that every system (K, R, $\oplus$, $\otimes$, $\oslash$) which satisfies the conditions expressed in P_1-P_{27}, below, is of the same type as the system of points in the plane.

P_1 : If a and b are elements of K, $a\oplus b$ is an element of K, called the *sum* of the elements a and b.

P_2 : $a\oplus b=b\oplus a$.

P_3 : $(a\oplus b)\oplus c=a\oplus(b\oplus c)$.

P_4 : If $a\oplus b=a\oplus b'$, then $b=b'$

P_5 : There is an element z in K such that $z\oplus z=z$.

Definition : If there is only one such element z, this unique element is called the *zero* element of the system.

P_6 : For every element a in K there is an element a' in K, such that $a\oplus a'=z$, where a is the zero element.

Definition : If this element a' is uniquely determined by a, it is called the *opposite of a* and is denoted by $-a$.

Any system (K, $\oplus$) that satisfies these P_1-P_6 is called an *Abelian Group* with respect to the operation. $\oplus$.

P_7 : If a and b are elements of K, then $a\otimes b$ is an element of K, called the *product* of the elements a and b.

P_8 : $a\otimes b=b\otimes a$.

P_9 : $(a\otimes b)\otimes c=a\otimes(b\otimes c)$.

P_{10} : If $a\otimes b=a\otimes b'$, and a is not zero, then $b=b'$.

P_{11} : $a\otimes(b\oplus c)=(a\otimes b)\oplus(a\otimes c)$.

P_{12} : There is an element u in K, different from zero, such that $u\otimes u=u$.

Definition : If there is only one such element u, this unique element is called the *unit* element of the system.

P_{13} : For every element a in K, provided a is not zero, there is an element a' in K, such that

$$a\otimes a'=u,$$

where u is the unit-element.

Definition: If this element a' is uniquely determined by a, it is called the *reciprocal* of a, and is denoted by $1/a$ or a^{-1} (provided a is not zero).

Any system(K, $\oplus$ $\otimes$) that satisfies these p_1-p_{13} is called a *field* with respect to the operations $\oplus$ and. $\otimes$,

The following propositions concern the class R and the relation ⧀:

P_{14}: If a and b are elements of R, and a not equal to b, then either $a ⧀ b$ or else $b ⧀ a$.

P_{15}: If $a ⧀ b$, then a is not equal to b.

P_{16}: If $a ⧀ b$, and $b ⧀ c$ then $a ⧀ c$.

These three propositions, P_{14}-P_{16}, make the class R an ordered class, with respect to the relation ⧀.

P_{17}: (*Dedekind's postulate*). If M is any (non-empty) subclass in R, and if there is an element c in R such that $a ⧀ c$ for every element a in M, then there is an element x in R having the following properties with regard to the subclass M:

i) if a belong to M, then $a ⧀ x$, or at most, $a = x$.

ii) if x' is any element of R such that $x' ⧀ x$, then there is at least one element a in M such that $x' ⧀ a$.

Definition: If this element x is uniquely determined by the subclass M, it is called the *upper limit of M*.

The following two propositions serve to connect the relation ⧀ with the operations $\oplus$ and. ⧀.

P_{18}: Within the class R, if $x ⧀ y$, then $a \oplus x ⧀ a \oplus y$.

P_{19}: Within the class R, if $z ⧀ a$ and $z ⧀ b$, where z is the zeroelement, then $z ⧀ a \otimes b$.

If, in P_{14}-P_{19}, we replace "R" by "K", then P_1-P_{19} form a complete of propositions for the *algebr of all real quantities*.

The following propositions concern the class R and the operations $\oplus$ and $\otimes$;

P_{20}: If a is an element of R,then a is an element of K.

P_{21}: The class R contains at least two elements.

P_{22}: If a and b belong to R, and have a sun $a \oplus b$, then $a \oplus b$ also belong to R,

P_{23}: If a belongs to R, and has an opposite, $-a$, then $-a$ also belongs to R.

P_{24}: If a and b belong to R, and have a product, $a \otimes b$, then $a \otimes b$ belongs to R.

P_{25}: If a belongs to R, and has a reciprocal, $1/a$, then $1/a$ also belongs to R.

These six propositions, $P_{20}-P_{25}$, together with P_1-P_{13}, make the class R, like the class K,a *field* with respect to $\oplus$ and $\otimes$.

P_{26}: If K is a field there is an element j in K such that $j \otimes j = -u$

where —u is the opposite of the unit element.

Definition: If there are two (and only two) such elements, j and —j, either of them may be called the *imaginary unit* of the system.

P_{27}: If K and R are field and K contains an imaginary unit j, then for every element a in K there are elements x and y in R, such that $x \oplus (j \otimes y) = a$.

These P_1-P_{27} form a complete set of propositions for *the algebra of complex quantities*. From these propositions all the theorems of the algebra of complex quantities can be deduced.

I want to state that these propositions are not by any means intended for use in elmentary instruction. Such a set of propositions exhibits the logical structure of a particular type of algebra. But I think an interest in the logical structure of a science naturally does not arise in a student's mind until the facts of that science have long been familiar to him.

It must not be supposed, moreover, that the set of pro-positions here given is the only possible set of propositions for the algebra in question: or that fundamental notions here mentioned are the ones that ale necessarily adopted.

§ 9. Sufficiency of the propositions.

P_1-P_{27} are sufficient to determine a definite type among the systems (K, R, ⊕, ⊗, ⧀); that is, any two systems (K, R, ⊕, ⊗, ⧀); that satisfy all these propositions will be *isomorphic* with respect to K, R, ⊕, ⊗, and ⧀.

proof: Suppose two systems (K, R, ⊕, ⊗, ⧀) and (K′, R′, ⊕′, ⊗′ ⧀′) are given.

First, pair the elements z and u of class K with the elements *z′* and *u′* of class K′; then pair all the rational real elements of K with the corresponding rational real elements of K′; and, further, pair the irra'ional real elements of K with the irrational real elments of K′ by pairing the limit of every sequence of rationals in K with the limit of the corresponding sequence of rationals in K′.

In this way a one-to-one correaspondence is established between the subclasses R and R′.

Next, taking one of the elements $\pm\sqrt{-u}$ in K as j, and one of the elements $\pm\sqrt{-u'}$ in K′ as *j′*, pair these elements *j* and *j′*; and finally pair every element x⊕j⊗y in K with the corresponing element *x′* ⊕′ *j′* ⊗′ *y′* in K′, thus completing the one-to-one correspondence between the two classes. It is then easy to see that the correspondence is of such a nature that if *a* and *b′* in K correspond to *a′* and *b′* in K′, then *a*⊕*b* will coresond to *a′*⊕*b′* and *a*⊗*b* to *a′* ⊗′ *b*, and, furthermore, if *a* ⧀*b*, then *a′*⧀′*b′*.

Namely,

(K, R, ⊕, ⊗, ⧀)		(K′, R′, ⊕′, ⊗′, ⧀′)
z ε K	←——→	*z′* ε K′
u ε K	←——→	*u′* ε K′
rational *u* ε K	←——→	rational *u′* ε K′
irrational *u* ε K	←——→	irrational *u′* ε K′
j ε K	←——→	*j′* ε K′
x⊕*j*×*y* ε K	←——→	*x′* ⊕′ *j′* ⊗′ y′ K′
a, *b* ε K		*a′*, *b′* ε K′
a	←——→	*a′*
b	←——→	*b′*
a ⊕ *b*	←——→	*a′* ⊕′ *b′*
a ⊗ *b*	←——→	*a′* ⊗′ *b′*
a ⧀ *b*	←——→	*a′* ⧀′ *b′*

The isomorphism between the two system is thus established.

Example 1: K=class of all points in the complex plane.

$a \oplus b = (a \times b)/(a+b)$, except that $a \oplus b = a+b$ whenever *a* or *b* is zero.

$a \otimes b = a \times b$

R=class of all points on the axis of reals.

$a \textcircled{<} b = a < b$, except that a and b are both positive or both negative, $(a \textcircled{<} b) = (a > b)$.

Here $z=0$, $u=1$, $j=i$.

Example 2: K=class of all points in the complex plane.

$a \oplus b = a + b$.

$a \otimes b = 5(a \times b)$

R=class of all points on the axis of reals.

$\textcircled{<} = <$.

Here $z=-0$, $u=1/5$, $j=i/5$.

Example 3: K=class of all points in the complane plane.

$a \oplus b = a + b + 1$

$a \otimes b = (a \times b) + (a + b)$

R=class of all points on the axis of reals.

$\textcircled{<} = <$.

ee $z=-1$, $u=0$, $-u=-2$, and $j=i-1$.

Each of these systems (Example 1—3) satisfies all the propositions of P_1-P_{27}, and hence is stirictly isomorphic with the system described in Part II*.

It will be noticed that the ordinary meaning of multiplication is preserved in Example 1, and the ordinary meaning of addition in Example 2.

§ 10 Examples of systems that satisfy all but one of the propositions.

The twenty-seven propositions of § 8 are all independent, that is, no one of them can be deduced from the remaining twenty-six.

To prove this, I want to exhibit, in the case of each proposition, a system(K, R, $\oplus$, $\otimes$, $\textcircled{<}$) which satisfies all the other propositions but not the one in question.

Here is a few examples.

Example for P_1. K=a class consisting of five elements, $0, 1, -1, i, -i$

R=a class consisting of three of these 5 elements, namely, $0, 1, -1$.

$\oplus = \otimes \qquad \times = \times \qquad \textcircled{<} = <$.

(*Note*) This system does not satisfy P_1, since, for example, the element $1+1=2$ does not belong to the class.

All the other propositions are satisfied

Example for P_3. K=all complex quantities. R=all real quantities. $a \oplus b = (a+b)/3$

$\otimes = \times \qquad \textcircled{<} = <$.

(*Note*) $(a \oplus b) \oplus c = \left(\frac{a+b}{3}\right) \oplus c = \left(\frac{a+b}{3} + c\right)/3 \qquad a \oplus (b \oplus c) = a \oplus \frac{b+c}{3} = \left(a + \frac{b+c}{3}\right)/3$

$$\therefore \ (a \oplus c) \oplus c \neq a \oplus (b \oplus c)$$

All the other propositions are satisfied.

Example for P_4. K=all complex quantities. R=all real quantities.

* Each of these system is obtained from the ordinary complex plane by a projective transformation.

$a \oplus b = 0$ for all value of a and b. $\otimes = \times$ $\oslash = <$

(*Note*) In this system, the law of cancellation for addition, is clearly not satisfied. There is a zero element $z=0$, and unit element $u=1$; and P_6 is satisfied. Since $a+a'=0$, whatever the value of a', we cannot speak of the opposite of a, since this element a' is not uniquely determined.

Hence propositions like P_{26} and P_{27}, which presuppouse the existence of an opposite, are satisfied vacuously.

Example for P_8. K=all complex quantities. R=all real quantities. $\oplus = +$ $a \otimes b = b$ $\oslash = <$

(*Note*) This system clearly does not satisfy the commutative law for multiplication.

All the other propositions are satisfied. (The system does not contain a unique unit element, and therefore all the propositions which presuppose such an element are satisfied vacuously.)

Example for P_{11}. K=all complex quantities. R=all real quantities. $\oplus = +$,

$a \otimes b = a+b-1$. $\oslash = <$.

(*Note*) Here $z=0$, $u=1$; since the distributive law is not satisfied, the system is not a field, and P_{26} and P_{27} demand nothing. All the other propositions are satisfied.

Example for P_{12}. K=the class of all complex quantities $x+iy$ in whih x and y are even integers.

R=all the elements of this class which are real

$\oplus = +$ $\otimes = \times$ $\oslash = <$.

(*Note*) This system contains no unit element, but satisfies all the other conditions.

Example for P_{16}. K=all complex quantities. R=all real real quantities. $\oplus = +$ $\otimes = \times$

$\oslash$=interpreted to mean "not equal to."

(*Note*) This system satisfies all the propositions except the law of transitivity; for, with the meaning given to $\oslash$, we may have $a \oslash b$ and $b \oslash c$, and yet not $a \oslash c$,

Example for P_{20}. K=all complex quantities $x+iy$, where x and y are restricted to rational values.

R=all real quantities. $\oplus = +$ $\otimes = \times$ $\oslash = <$

(*Note*) This system satisfies all the propositions except P_{20}.

Example for P_{24}. K=all complex quantities, with $\oplus = +$, and $\otimes = \times$,

R=all pur imaginaries, with $\oslash$ defined so that $ix \oslash iy$ whenever $x<y$.

(*Note*) Here the product of two elements of R will not (in general) belong to R, but all the other propositions are satisfied. (P_{19} and P_{27} vacuoualy).

Example for P_{26}. K=all real quantities. R=all real quantities. $\oplus = +$ $\otimes = \times$ $\oslash = <$

(*Note*) This system contains no imaginary units.

IV. CONCLUSION

I think I am now in a position to answer the following question.

1 *st Question*: "What is the algebra of complex quantities?" The answer is this:

The algebra of complex quantities is the scientific study of that particular type of system (K, R, $\oplus$, $\otimes$, $\oslash$)" which satisfies the twenty-seven primitive propositions of § 8. Namely, any system (K, R, $\oplus$, $\otimes$, $\oslash$) that satisfies these P_1-P_{27} conditions may be taken as a representative example of the algebra, and all the propositions which are logically deducible from these tweenty-seven Primitive propositions are the popositions which from the body of the science.

The system of points in a plane, described in part II, is the simplest representative example of this algebra, and is the only example which could possibly be used to advantage in elementary instruction.

2 nd Question: " What is an imaginary quantity ? " The answer is this:

If any system (K, R, ⊕, ⊗, ⊘) that satisfies the twenty-seven laws of complex algebra is given, then any element of K, not belonging to the subcllass R, is called an imaginary element of that system.

Among the twenty-seven conditions § 8, if we omit P_{20}-P_{27}, and abandon the distinction between the classes K and R, and make P_{14}-P_{19} apply to the whole class K, then P_1-P_{19} will be satisfied the system (K, ⊕, ⊗, ⊘) which is called the *Algebra of all Real Quantities.*

Complete sets of conditions for other subalgebras, as the algebr of positive integers, the algebra of all intesgers, the algebra of all rationals, etc., are more complicated, that is, more subject to exceptions, than are the rules of operation in the general algebra of complex quantities. On this account, it is usually worth while to employ the algebra of complex quantities even in cases where the data of the problem, and the required answer, are all real quanties.

For example, if it is required to find a real value of x that satisfies a given equation

$$ax_2+bx+c=0,$$

the simplest plan is first to find all the values of x that satisfy the equation, and then to pick out those, if any, that are real. Similarly, if the problem calls for a positive value (or an integral value) of x, we do not confine ourselves to the algebra of positive quantities(or the algebra of integral quantities) but proceed at once to operate in the realm of all complex quantites and then select those result that satisfy the given condition.

It is chiefly for reasons of this sort that the algebra of complex quantities shoud be taught in the secondary schools, so that most students can solve some elementary practical problems in which this type of algebra is directly applicable. However, it will be obvious that these arithmetical systems are wholly unsuitable for use in elementary instruction. And I intend to speak of the elements of such an arithmetrical system as the genuine *algebraic quantities*, and to regard the points in the plane as merely geometrical representations of them. (April 13, 1967)

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