ON THE HYDROFOIL OF FINITE ASPECT RATIO MOVING AT FINITE FROUDE NUMBER

By

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I. INTRODUCTION

In this paper the method of determining the hydrodynamic forces exerted on a nonplanar hydrofoil of finite aspect ratio undergoing a steady rectilinear motion at a constant submergence is studied.

Applying the lifting-line theory Wu(1)**, Nishiyama(2), Kaplan, et al(3), Breslin(4) and Isay(5) investigated the hydrodynamic properties of a foil of large aspect ratio. Recently, Ashley(6) and Widnall(7) treated hydroelastic problems of a foil of finite aspect ratio at large Froude number using a kernel function procedure developed by Watkins, et al(8), which is based on the lifting-surface theory.

The linearized boundary value problem associated with the lifting surface is usually formulated in terms of the velocity potential. Then, the kinematic condition must be satisfied over the lifting surface as well as its wake. However, in the kernel function procedure by Watkins, et al, the use of Prandtl's acceleration potential permits us to relate the discontinuity of the potential to the prescribed normal-wash only over the lifting surface. The resulting basic equation is a Fredholm integral equation of the first kind containing a second order singularity in the kernel. The definition of the improper integral involving such singularity is given by Mangler(9). Therefore, representing the discontinuity of acceleration potential on the foil by a series, the coefficients of the series can be determined by solving the basic equation through a numerical procedure.

Upon formulating the boundary value problem of the hydrofoil, the Green's function is determined by satisfying the linearized free surface condition and the condition at infinity. This Green's function is valid for a nonplanar foil moving at finite Froude number. It

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reduces to a simpler form for a planar foil. In the case of very small or very large Froude number, the Green's function can be further simplified to contain only the fundamental singularity (i.e., the singularity in a fluid of infinite extent given as the local exterior normal derivative of the inverse distance between a point in space and the points on the lifting surface) and its negatively or positively reflected image with respect to the undisturbed free surface. In passing, it should be noted that the principal value integral in the Green's function is transformed into a more tractable form using the exponential integral.

Next, we proceed to show the formulation of the integral equation by use of a continuous distribution of the acceleration doublet which produces the velocity field of horseshoe vortex. Then the method of solution which is based on Watkin's procedure is presented after the discussion of the singularity in the kernel of the equation. In the last section, the numerical problem is described by introducing suitable coordinates for a foil of an elliptic cross section. The formulae for determining the lift, induced drag and moment on the hydrofoil when the coefficient of series are known are also included.

II. FORMULATION OF THE PROBLEM

Let us consider a thin lifting surface travelling at a uniform speed U underneath a free surface of an incompressible fluid. Denoting the undisturbed free surface by $z=0$, we take the x-axis positive in the direction of the motion. Then, following the usual linearization of thin airfoil theory, the boundary value problem can be stated as follows. Find the induced velocity potential $\phi(x,y,z)$ which satisfies the following conditions:

$$\left.\begin{array}{lll} \text{(A)} & \nabla^2\phi(x,y,z)=0 & \text{in } z<0 \\ \text{(B)} & \dfrac{\partial^2}{\partial x^2}\phi(x,y,0)+\nu\dfrac{\partial}{\partial z}\phi(x,y,0)=0 & \text{on the free surface,} \\ \text{(C)} & \dfrac{\partial}{\partial n}\phi(x,y,z)=-U\dfrac{\partial x}{\partial n} & \text{on the lifting surface,} \\ \text{(D)} & \lim\limits_{x\to\infty}\phi(x,y,z)=0 & \text{far ahead of the lifting surface,} \end{array}\right\} \quad (1)$$

where $\nu=g/U^2$. Therefore, the Froude number is given by $F=\dfrac{1}{\sqrt{\nu L}}$ (L being a typical length). Note that the x, y, z coordinates are fixed in the lifting surface.

From Bernoulli's equation

$$p+\frac{\rho}{2}\left[\left(U+\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right]=p_0+\frac{\rho}{2}U^2,$$

neglecting the squared terms, the relative pressure is given by

$$\frac{p-p_0}{\rho}=U\frac{\partial\phi}{\partial x}=\Omega(x,y,z) \quad (2)$$

However, according to Euler's equation, i.e.

$$\frac{Dv}{Dt} = -\ \text{grad}\ \frac{p}{\rho},$$

grad Ω may be considered as the approximation for the acceleration vector. Therefore, it is called an induced acceleration potential.

For example, the fundamental singularity for an airfoil (acceleration doublet) moving in a fluid of unbounded extent is given by

$$\Omega_0(x,y,z;\xi,\eta,\zeta) = \frac{\partial}{\partial n'}\frac{1}{R} \quad , \qquad R^2 = (x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2 \quad , \tag{3}$$

where n' is the exterior normal at a point (ξ,η,ζ) on the lifting surface. Then, the singularity for the induced velocity potential (horseshoe vortex) is

$$\phi_0(x,y,z;\xi,\eta,\zeta) = \frac{1}{U}\int_{\infty}^{x}\frac{\partial}{\partial n'}\frac{1}{R}\Bigg|_{x=\lambda} d\lambda. \tag{4}$$

where $\frac{\partial}{\partial n'} \equiv n'_\eta \frac{\partial}{\partial \eta} + n'_\zeta \frac{\partial}{\partial \zeta}$.

The Green's function G which satisfies the given set of the boundary condition (1 − A, B and D) may be found by setting

$$G(x,y,z;\xi,\eta,\zeta) = \phi_0(x,y,z;\xi,\eta,\zeta) + H(x,y,z;\xi,\eta,\zeta). \tag{5}$$

As the function G not only depends on the parameter ν but also on the geometry of the surface, we choose the foil to be a surface having a spanwise curvature (see Figure 1). Thus, (4) can be rewritten as

$$\begin{aligned}\phi_0(x,y,z;\xi,\eta,\zeta) &= -\frac{1}{U}\int_{\infty}^{x}\left(n'_\zeta \frac{\partial}{\partial z} + n'_\eta \frac{\partial}{\partial y}\right)\frac{d\lambda}{\sqrt{(\lambda-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}} \\ &= \frac{1}{U}\left[n'_\zeta \frac{z_0}{y_0^2 + z_0^2} + n'_\eta \frac{y_0}{y_0^2 + z_0^2}\right]\left[\frac{x_0}{\sqrt{x_0^2 + y_0^2 + z_0^2}} - 1\right],\end{aligned} \tag{6}$$

where $x_0 = x-\xi$, $y_0 = y-\eta$, $z_0 = z-\zeta$, $n'_\eta = \frac{\partial \eta}{\partial n}$, and $n'_\zeta = \frac{\partial \zeta}{\partial n'}$

Further, making use of the following identities; for $z>0$,

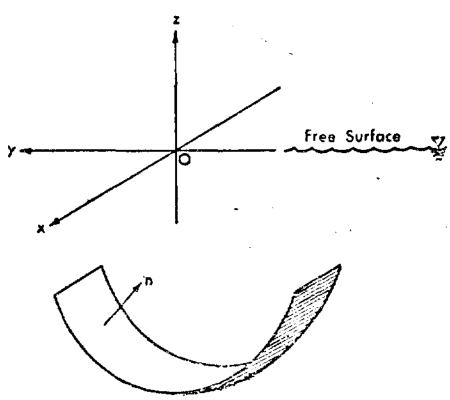

Figure 1. Submerged Lifting Surface with a Spanwise Curvature.

$$\left.\begin{aligned}&\int_0^\infty e^{-\mu z}\cos\mu y d\mu=\frac{z}{y^2+z^2},\\&\int_0^\infty e^{-\mu z}\sin\mu y d\mu=\frac{y}{y^2+z^2}\\&\frac{2}{\pi}\int_0^\infty\frac{\sin\lambda x}{\lambda}d\lambda\int_0^\infty e^{-\sqrt{\mu^2+\lambda^2}\,z}\cos\mu y d\mu=\frac{z}{y^2+z^2}\frac{x}{\sqrt{x^2+y^2+z^2}}\\&\frac{2}{\pi}\int_0^\infty\frac{\sin\lambda x}{\lambda}d\lambda\int_0^\infty\frac{\mu}{\sqrt{\mu^2+\lambda^2}}e^{-\sqrt{\mu^2+\lambda^2}\,z}\sin\mu y d\mu=\frac{y}{y^2+z^2}\frac{x}{\sqrt{x^2+y^2+z^2}}\end{aligned}\right\}\tag{7}$$

we find

$$\phi_0(x,y,z;\xi,\eta,\zeta)=\frac{n'_\zeta}{U}\int_0^\infty\left[\frac{2}{\pi}\int_0^\infty e^{-\sqrt{\mu^2+\lambda^2}\,z_0}\frac{\sin\lambda x_0}{\lambda}d\lambda-e^{-\mu z_0}\right]\cos\mu y_0 d\mu$$
$$+\frac{n'_\eta}{U}\int_0^\infty\left[\frac{2}{\pi}\int_0^\infty e^{-\sqrt{\mu^2+\lambda^2}\,z_0}\frac{\mu\sin\lambda x_0}{\lambda\sqrt{\mu^2+\lambda^2}}d\lambda-e^{-\mu z_0}\right]\sin\mu y_0 d\mu.\tag{8}$$

If the variables in Eq. (8) are altered by $\lambda=k\cos\theta$, $\mu=k\sin\theta$, ϕ_0 can be transformed in terms of the exponential function as

$$\phi_0(x,y,z;\xi,n,\zeta)=-\frac{n'_\zeta}{U}\left[Re\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^\infty ie^{-kz_0+ik\omega}\sec\theta dkd\theta+\int_0^\infty e^{-\mu z_0}\cos\mu y_0 d\mu\right]$$
$$-\frac{n'_\eta}{U}\left[Re\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^\infty e^{-kz_0+ik\omega}\tan\theta dkd\theta+\int_0^\infty e^{-\mu z_0}\sin\mu y_0 d\mu\right],\tag{9}$$

where $\omega=(x-\xi)\cos\theta+(y-\eta)\sin\theta$.

Observing here that all the integrands in Eq. (9) satisfy condition (1—A), we may assume the function H in Eq. (5) in the form

$$H(x,y,z;\xi,\eta,\zeta)=-Re\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^\infty f(k,\theta)e^{k\rho+ik\omega}\left(i\frac{n'_\zeta}{U}\sec\theta+\frac{n'_\eta}{U}\tan\theta\right)dkd\theta$$
$$-\int_0^\infty e^{\mu\rho}\left(\frac{n'_\zeta}{U}\cos\mu y_0+\frac{n'_\eta}{U}\sin\mu y_0\right)d\mu,\tag{10}$$

where $\rho=-|z+\zeta|<0$. Then application of condition (1—B) to the sum ϕ_0+H yields

$$f(k,\theta)=-\left(1+\frac{2\nu\sec^2\theta}{k-\nu\sec^2\theta}\right),\tag{11}$$

Therefore, by (5) the Green's function becomes

$$G(x,y,z;\xi,\eta,\zeta)=-\frac{n'_\zeta}{U}\left\{Re\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^\infty i(e^{-2kz}-1)e^{k(\rho+i\omega)}\sec\theta d\theta dk\right.$$
$$\left.+\int_0^\infty(e^{-2\mu z}+1)e^{\mu\rho}\cos\mu y_0 d\mu-Re\left[P.V.\frac{\nu}{\pi}\int_{-\pi}^{\pi}\int_0^\infty\frac{ie^{k(\rho+i\omega)}}{k-\nu\mathrm{se}^2\theta}\sec^3\theta dkd\theta\right]\right\}$$
$$-\frac{n'_\eta}{U}\left\{Re\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^\infty(e^{-2kz}-1)e^{k(\rho+i\omega)}\tan\theta dkd\theta+\int_0^\infty(e^{-2\mu z}+1)\right.$$
$$\left.\times e^{\mu\rho}\sin\mu y_0 d\mu-Re\left[P.V.\frac{\nu}{\pi}\int_{-\pi}^{\pi}\int_0^\infty\frac{e^{k(\rho+i\omega)}}{k-\nu\sec^2\theta}\tan\theta\sec^2\theta dkd\theta\right]\right\}.\tag{12}$$

Here, the first and second integrals within curly brackets represent the original singularity given by (6) and its reflected image with respect to the undisturbed free surface, $z=0$ plane. Far ahead of the foil, for $x=+\infty$, these integrals immediately satisfy condition (1—D). Next, the remaining principal-value integral is to be examined.

It is known that the principal-value integral of a complex argument $\tau=\rho+i\omega$, and a parameter $\gamma=\nu\sec^2\theta$ can be transformed into a definite integral by (see Refs. 10 or 11):

$$P.V.\int_0^\infty \frac{e^{k\tau}}{k-\gamma}\,dk = e^{\gamma\tau}E_1(\gamma\tau)+R^*(\gamma\tau), \tag{13}$$

where

$$E_1(\gamma\tau)=\int_{\gamma\tau}^\infty \frac{e^{-t}}{t}\,dt = e^{-i\gamma\omega}\int_{\gamma\rho}^\infty \frac{e^{-m}}{m+i\gamma\omega}\,dm,$$

and

$$R^*(\gamma\tau)=\begin{cases} i2\pi e^{\gamma\tau} & \text{over } \frac{\pi}{2}\leqq\theta\leqq\frac{\pi}{2}+\alpha,\\ -i2\pi e^{\gamma\tau} & \text{over } -\frac{\pi}{2}\leqq\theta\leqq-\frac{\pi}{2}+\alpha, \text{ with } \alpha=\text{arc tan}\left(\frac{y-\eta}{x-\xi}\right). \end{cases}$$

Hence, by the identity (13), the principal-value integrals in (12) can be expressed in a more tractable from as

$$\left.\begin{aligned} &Re\left[P.V.\frac{\nu}{\pi}\int_{-\pi}^{\pi}\int_0^\infty \frac{ie^{k(\rho+i\omega)}}{k-\nu\sec^2\theta}\sec^3\theta\,dkd\theta\right]\\ &\quad=-\frac{\nu}{\pi}\int_{-\pi}^{\pi} Im\left[e^{\nu\sec^2\theta(\rho+i\omega)}E_1[\nu\sec^2\theta(\rho+i\omega)]\right]\sec^3\theta d\theta\\ &\qquad-4\nu\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\alpha} e^{\nu\sec^2\theta\rho}\cos(\nu\sec^2\theta\omega)\times\sec^3\theta d\theta,\\ &\text{and}\\ &Re\left[P.V.\frac{\nu}{\pi}\int_{-\pi}^{\pi}\int_0^\infty \frac{e^{k(\rho+i\omega)}}{k-\nu\sec^2\theta}\tan\theta\sec^2\theta dkd\theta\right.\\ &=\frac{\nu}{\pi}\int_{-\pi}^{\pi} Re\left[e^{\nu\sec^2\theta(\rho+i\omega)}E_1[\nu\sec^2\theta(\rho+i\omega)]\right]\tan\theta\sec^2\theta d\theta-4\nu\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\alpha}\nu\sec^2\theta\rho\\ &\qquad\qquad\times\sin(\nu\sec^2\theta\omega)\tan\theta\sec^2\theta d\theta \end{aligned}\right\} \tag{14}$$

For a large argument, the exponential integral E_1 can be approximated by

$$E_1[\nu\sec^2\theta(\rho+i\omega)]\sim\frac{e^{-\nu\sec^2\theta(\rho+i\omega)}}{\nu\sec^2\theta(\rho+i\omega)},$$

$$\text{hence,}\quad Re\left[e^{\nu\sec^2\theta(\rho+i\omega)}E_1[\nu\sec^2\theta(\rho+i\omega)]\right]\sim\frac{\rho}{\nu\sec^2\theta(\rho^2+\omega^2)},$$

$$\text{and}\quad Im\left[e^{\nu\sec^2\theta(\rho+i\omega)}E_1[\nu\sec^2\theta(\rho+i\omega)]\right]\sim-\frac{\omega}{\nu\sec^2\theta(\rho^2+\omega^2)}.$$

Therefore, the first integrals in Eq. (14) vanish as x tends to infinity. Here the second integrals also become zero since $\alpha=0$ for $x=\infty$. Thus, it can be seen that the principal-value integral satisfies condition (1−D).

Finally, substituting (14) into (12), we obtain

$$G(x,y,z;\xi,\eta,\zeta)=\frac{n'_\zeta}{U}\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^\infty(e^{-2kz}-1)e^{k\rho}\sin(k\omega)\sec\theta dkd\theta-\int_0^\infty(e^{-2\mu z}+1)e^{\mu\rho}\cos\mu y_0 d\mu\right.$$

$$-\frac{\nu}{\pi}\int_{-\pi}^{\pi} Im\left[e^{\nu\sec^2\theta(\rho+i\omega)}E_1[\nu\sec^2\theta(\rho+i\omega)]\right]\sec^3\theta d\theta$$

$$-4\nu\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\alpha} e^{\nu\sec^2\theta\rho}\cos(\nu\sec^2\theta\omega)\sec^3\theta d\theta\Big\}$$

$$+\frac{n'_\eta}{U}\Big\{-\frac{1}{2\pi}\int_{-\pi}^{\pi}\int_0^\infty(e^{-2kz}-1)e^{k\rho}\cos(k\omega)\tan\theta dk d\theta-\int_0^\infty(e^{-2\mu z}+1)e^{\mu\rho}\sin\mu y_0 d\mu$$

$$+\frac{\nu}{\pi}\int_{-\pi}^{\pi} Re\left[e^{\nu\sec^2\theta(\rho+i\omega)}E_1[\nu\sec^2\theta(\rho+i\omega)]\right]\tan\theta\sec^2\theta d\theta$$

$$-4\nu\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\alpha} e^{\nu\sec^2\theta\rho}\sin(\nu\sec^2\theta\omega)\tan\theta\sec^2\theta d\theta\Big\} \qquad (15)$$

where $\rho=z+\zeta$, $\omega=(x-\xi)\cos\theta+(y-\eta)\sin\theta$, $\alpha=\arctan\left(\frac{y-\eta}{x-\xi}\right)$

III. INTEGRAL REPRESENTATION

Suppose the lifting-surface is represented by a continuous distribution of the doublet, namely the acceleration potential Ω, than at apoint (x, y, z) in the fluid region we have

$$\Psi(x, y, z)=\frac{1}{4\pi}\iint_S f(\xi, \eta, \zeta)\Omega(x, y, z; \xi, \eta, \zeta)dS. \qquad (16)$$

Therefore, from (4) the corresponding induced velocity potential can be expressed in the form

$$\phi(x, y, z)=\frac{1}{4\pi}\iint_S f(\xi, \eta, \zeta)G(x, y, z; \xi, \eta, \zeta)dS, \qquad (17)$$

where $$G(x, y, z; \xi, \eta, \zeta)=\frac{1}{U}\int_\infty^x \Omega(\lambda, y, z; \xi, \eta, \zeta)d\lambda,$$

and f denotes the strength of the distributed singularity.

According to the potential theory, the strength of the doublet is locally proportioned to the discontinuity of the potential Ψ, that is

$$f(\xi, \eta, \zeta)=\Psi_u(\zeta, \eta, \zeta)-\Psi_l(\xi, \eta, \zeta) \qquad (18)$$

Thus, from (2) the pressure jump Δp at any point on the lifting surface can be related to the strength of the singularity by

$$\Delta p(\xi, \eta, \zeta)=P_l(\xi, \eta, \zeta)-P_u(\xi, \eta, \zeta)=\rho f(\xi, \eta, \zeta). \qquad (19)$$

If the strength f is determined from the integral equation

$$\frac{1}{4\pi U}\iint_S f(\xi, \eta, \zeta)\frac{\partial}{\partial n}G(x, y, z; \xi, \eta, \zeta)dS=-\frac{\partial x}{\partial n} \qquad (20)$$

the induced velocity potential (17) is the solution of the given boundary value problem since it satisfies condition (1−C). Once the value of f is known, the force and moment acting on the lifting surface can be determined by

$$\left.\begin{aligned} \underset{\sim}{F} &= \iint_S \Delta p \, \underset{\sim}{n} \, ds = \rho \iint_S f(\xi, \eta, \zeta) \, \underset{\sim}{n} dS \\ \underset{\sim}{M} &= \iint_S \Delta p (\underset{\sim}{r} \times \underset{\sim}{n}) dS = \rho \iint_S f(\xi, \eta, \zeta) (\underset{\sim}{r} \times \underset{\sim}{n}) dS \end{aligned}\right\} \tag{21}$$

The kernel of the integral equation (20) is the normal derivative of the function G given by (12) and that is

$$\begin{aligned} \frac{\partial}{\partial n} G = \frac{n'_\zeta}{U} &\left\{ \frac{\partial}{\partial n} \left[\frac{z_0}{r^2} \left(\frac{x_0}{R} - 1 \right) - \frac{\rho}{r'^2} \left(\frac{x_0}{R'} - 1 \right) \right] \right. \\ &\left. + Re \left[P.V. \frac{\nu}{\pi} \int_{-\pi}^{\pi} \int_0^\infty \frac{\partial}{\partial n} \frac{i e^{k(\rho + i\omega)}}{k - \nu \sec^2 \theta} \sec^3 \theta dk d\theta \right] \right\} \\ &+ \frac{n'_\eta}{U} \left\{ \frac{\partial}{\partial n} \left[\frac{y_0}{r^2} \left(\frac{x_0}{R} - 1 \right) - \frac{y_0}{r'^2} \left(\frac{x_0}{R'} - 1 \right) \right] \right. \\ &\left. + Re \left[P.V. \frac{\nu}{\pi} \int_{-\pi}^{\pi} \int_0^\infty \frac{\partial}{\partial n} \frac{e^{k(\rho + i\omega)}}{k - \nu \sec^2 \theta} \tan \theta \sec^2 \theta \, dk d\theta \right] \right\} \end{aligned} \tag{22}$$

where $x_0 = x - \xi$, $y_0 = y - \eta$, $z_0 = z - \zeta$, $\rho = z + \zeta$, $\omega = x_0 \cos \theta + y_0 \sin \theta$,

$r^2 = y^2_0 + z^2_0$, $r'^2 = y^2_0 + \rho^2$, $R^2 = x^2_0 + r^2$, and $R'^2 = x^2_0 + r'^2$.

However, making use of the identity (13) we find

$$\begin{aligned} P.V. &\int_0^\infty \frac{\partial}{\partial n} \frac{e^{k\tau}}{k - \gamma} dk = P.V. \int_0^\infty \frac{k e^{k\tau}}{k - \gamma} \left(n_z \frac{\partial}{\partial z} + n_y \frac{\partial}{\partial y} \right) \tau \, dk \\ &= \gamma (n_z + i \sin \theta \, n_y) \left[- \frac{1}{\gamma \tau} + e^{\gamma\tau} E_1(\gamma\tau) + R^*(\gamma\tau) \right] \\ &= \gamma (n_z + i \sin \theta \, n_y) \left[- \frac{\rho}{\gamma(\omega^2 + \rho^2)} + Re[e^{\gamma\tau} E_1(\gamma\tau)] - (\text{sign} \, \theta) 2 \pi e^{\gamma\rho} \sin(\gamma\omega) \Big|_{\theta = \pm \frac{\pi}{2}}^{\theta = \pm \frac{\pi}{2} + \alpha} \right] \\ &\quad - \gamma (\sin \theta \, n_y + i n_z) \left[- \frac{\omega}{\gamma(\omega^2 + \rho^2)} + [Im \, e^{\gamma\tau} E_1(\gamma\tau)] \right. \\ &\quad \left. - (\text{sign} \, \theta) 2 \pi e^{\gamma\rho} \cos(\gamma\omega) \Big|_{\theta = \pm \frac{\pi}{2}}^{\theta = \pm \frac{\pi}{2} + \alpha} \right] \end{aligned} \tag{23}$$

where $\tau = \rho + i\omega$, $\gamma = \nu \sec^2 \theta$, $Re[e^{\gamma\tau} E_1(\gamma\tau)] = e^{\gamma\rho} \int_{\gamma\rho}^\infty \frac{m e^{-m}}{m^2 + (\gamma\omega)^2} dm$,

and $$Im[e^{\gamma\tau} E_1(\gamma\tau)] = - e^{\gamma\rho} \int_{\gamma\rho}^\infty \frac{\gamma\omega e^{-m}}{m^2 + (\gamma\omega)^2} dm.$$

By carrying out the differentiation of the remaining part in (22) and applying the formular (23), the result can be expressed in the following form:

$$\begin{aligned} \frac{\partial}{\partial n} G = \frac{n'_\zeta n_z}{U} &\left[\frac{y^2_0 - z^2_0}{r^4} \left(\frac{x_0}{R} - 1 \right) - \frac{x_0 z^2_0}{rR} - \frac{y^2_0 - \rho^2}{r'^4} \left(\frac{x_0}{R'} - 1 \right) + \frac{x_0 \rho^2}{r'^2 R'^3} \right] \\ + \frac{n'_\eta n_z + n'_\zeta n_y}{U} &\left[- \frac{2 y_0 z_0}{r^4} \left(\frac{x_0}{R} - 1 \right) - \frac{x_0 y_0 z_0}{r^2 R^3} + \frac{2 y_0 \rho}{r'^4} \left(\frac{x_0}{R'} - 1 \right) + \frac{x_0 y_0 \rho}{r'^2 R'^3} \right] \\ + \frac{n'_\eta n_y}{U} &\left[- \frac{y^2_0 - z^2_0}{r^4} \left(\frac{x_0}{R} - 1 \right) - \frac{x_0 y^2_0}{r^2 R^3} + \frac{y^2_0 - \rho^2}{r'^4} \left(\frac{x_0}{R'} - 1 \right) + \frac{x_0 y^2_0}{r'^2 R'^3} \right] \end{aligned}$$

$$+\frac{n'_\zeta n_z}{U}\left\{\frac{\nu^2}{\pi}\int_{-\pi}^{\pi}\left[-\frac{\omega}{\nu\sec^2\theta(\rho^2+\omega^2)}+Im\left[e^{\nu\sec^2\theta(\rho+i\omega)}E_1[\nu sec^2\theta(\rho+i\omega)]\right]\right]\sec^5\theta\right.$$

$$\left.-4\nu^2\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\alpha}e^{\nu\sec^2\theta\rho}\cos(\nu\sec^2\theta\omega)\sec^5\theta d\theta\right\}$$

$$+\frac{n'_\eta n_z-n'_\zeta n_y}{U}\left\{\frac{\nu^2}{\pi}\int_{-\pi}^{\pi}\left[-\frac{\rho}{\nu\sec^2\theta(\rho^2+\omega^2)}+Re\left[e^{\nu\sec^2\theta(\rho+i\omega)}E_1[\nu\sec^2\theta(\rho+i\omega)]\right]\right]\right.$$

$$\left.\times\tan\theta\sec^4\theta\, d\theta-4\nu^2\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\alpha}e^{\nu\sec^2\theta\rho}\sin(\nu\sec^2\theta\omega)\tan\theta\sec^4\theta\, d\theta\right\}$$

$$-\frac{n'_\eta n_y}{U}\left\{\frac{\nu^2}{\pi}\int_{-\pi}^{\pi}\left[-\frac{\omega}{\nu\sec^2\theta(\rho^2+\omega^2)}+Im\left[e^{\nu\sec^2\theta(\rho+i\omega)}E_1[\nu\sec^2\theta(\rho+iw)]\right]\right]\right.$$

$$\left.\tan^2\theta\sec^3\theta\, d\theta-4\nu^2\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\alpha}e^{\nu\sec^2\theta\rho}\cos(\nu\sec^2\theta\omega)\tan^2\theta\sec^3\theta d\theta\right\} \qquad (24)$$

The first term of the kernel becomes singular along $y=\eta$ and $z=\zeta$, on the lifting surface. Therefore, we must examine the property of the kernel in the neighborhood of such chord σ which consists of a strip having the width 2ε. In this region σ, relating a small distance r to the normal and tangential coordinates by

$$r^2=(y-\eta)^2+(z-\zeta)^2=(n-m)^2+(t-s)^2$$

the singular term can be expressed as

$$\frac{\partial}{\partial n}\phi_0=\frac{1}{U}\int_{\infty}^{x}\frac{\partial^2}{\partial n^2}\frac{1}{\sqrt{(\lambda-\xi)^2+r^2}}d\lambda$$

$$=\frac{1}{U}\int_{\infty}^{x}\cdot\left[\frac{(t-s)^2}{r^3}\frac{\partial}{\partial r}+\frac{(n-m)^2}{r^2}\frac{\partial^2}{\partial r^2}\right]\frac{d\lambda}{\sqrt{(\lambda-\xi)^2+r^2}}$$

$$=\frac{1}{U}\left\{\left[\frac{1}{r^2}-\frac{3(n-m)^2}{r^4}\right]\left[1-\frac{x-\xi}{\sqrt{(x-\xi)^2+r^2}}\right]\right.$$

$$\left.+\frac{(n-m)^2}{r^4}\left[1-\frac{(x-\xi)^3}{[(x-\xi)^2+r^2]^{3/2}}\right]\right\}$$

$$\approx\frac{1}{U}\left\{\left[\frac{1}{r^2}-\frac{3(n-m)^2}{r^4}\right]\left(1-\frac{x-\xi}{|x-\xi|}\right)+\frac{(n-m)^2}{r^4}\left(1-\frac{x-\xi}{|x-\xi|}\right)\right\} \qquad (25)$$

According to Mangler's definition of the improper integral, i.e.

$$\int_a^b\frac{f(s)}{(t-s)^2}ds=\lim_{\varepsilon\to0}\int_a^{t-\varepsilon}+\int_{t+\varepsilon}^{b}\frac{f(s)}{(t-s)^2}ds-\frac{2}{\varepsilon}f(t), \qquad (26)$$

the integral of (25) appearing in (20) can now be approximated in the neighborhood region σ, and the remaining region $S-\sigma$ by

$$\frac{1}{4\pi U}\iint_S f(\xi,\eta,\zeta)\frac{\partial}{\partial n}\phi_0(x,y,z;\xi,\eta,\zeta)\,dS$$

$$\approx\frac{1}{4\pi U}\left\{\iint_{S-\sigma}f(\xi,\eta,\zeta)\frac{\partial}{\partial n}\phi_0\,dS+\lim_{m\to n}\frac{1}{U}\int_{-c}^{c}\int_{t-\varepsilon}^{t+\varepsilon}f(\xi,m,s)\right.$$

$$\times\left[\left(\frac{1}{r^2}-\frac{3(n-m)^2}{r^4}\right)\left(1-\text{sign}(x-\xi)\right)\right.$$
$$\left.+\frac{(n-m)^2}{r^4}\left(1-\text{sign}(x-\xi)\right)\right]ds d\xi\Bigg\}$$
$$=\frac{1}{4\pi U}\left[\iint_{S-\sigma} f(\xi,\eta,\zeta)\frac{\partial}{\partial n}\phi_0\,dS+\frac{1}{U}\int_{-c}^{c}\int_{t-\varepsilon}^{t+\varepsilon}\frac{f(\xi,\eta,s)}{(t-s)^2}\right.$$
$$\left.\times\left(1-\text{sign}(x-\xi)\right)ds d\xi\right] \tag{27}$$

The integral equation (20) can be solved approximately by means of reducing the given equation to a finite set of linear algebraic equations. To facilitate the solution, customarily the unknown function f is expanded in a series of chordwise and foilwise variables containing the undetermined coefficients a_{nm}, that is

$$f(\xi,\eta,\zeta)=\hat{f}(\delta,q)=U^2\sum_{n=0}^{N}\sum_{m=0}^{M}a_{nm}g_m(q)l_n(\delta) \tag{28}$$

$$\left.\begin{aligned}\text{with}\quad &\xi=m(q)+c(q)\cos\delta \qquad [0\leq\delta\leq\pi]\\ &g_m(q)=\left(\frac{q}{q_0}\right)^m\sqrt{1-\left(\frac{q}{q_0}\right)^2}\\ &l_0(\delta)=\cot\frac{\delta}{2},\ l_n(\delta)=\frac{4}{2^{2n}}\sin n\delta\end{aligned}\right\}$$

where q_0 is the half-foil length, and q is the coordinate along the arc of the foil so that $m(q)$ and $c(q)$ represent the equation of the midchord and the half-chord length, respectively. Here the symmetric loading about the root chord requires only even m while the anti-symmetric loading only odd m. For a practical application of the expansion (28), four chordwise and four foilwise lifting functions are to be used, hence $N=3$ and $M=3$. It is therefore necessary to choose the sixteen control points on the lifting surface for the determination of the coefficient a_{nm}. As shown by (27) the term $\frac{\partial}{\partial n}\phi_0$ possesses a second-order singularity, namely

$$\lim_{\substack{\eta\to y\\ \zeta\to z}}\frac{\partial}{\partial n}\phi_0=\lim_{s\to t}\frac{1}{(t-s)^2}[1-\text{sign}(x-\xi)]$$
$$\approx\lim_{q\to p}\frac{1}{(p-q)^2}[1-\text{sign}(\delta-\gamma)] \tag{29}$$

provided that the width 2ε of the neighborhood region σ is sufficiently small in comparison to the radius of foil curvature.

From (28) and (29), the integral equation (20) can be rewritten as

$$\frac{U}{4\pi}\sum_{n=0}^{N}\sum_{m=0}^{M}a_{nm}\left[\int_{p-\varepsilon}^{p+\varepsilon}\frac{g_m(q)}{(p-q)^2}dq\left(\int_0^{\gamma}+\int_{\gamma}^{\pi}\right)l_n(\delta)[1-\text{sign}(\delta-\gamma)]c(q)\sin\delta d\delta\right.$$
$$+\left(\int_{-q_0}^{p-\varepsilon}+\int_{p+\varepsilon}^{q_0}\right)\int_0^{\pi}g_m(q)l_n(\delta)\frac{\partial}{\partial n}\hat{\phi}_0(p,\gamma;q,\delta)c(q)\sin\delta d\delta dq$$
$$\left.+\int_{-q_0}^{q_0}\int_0^{\pi}g_m(q)l_n(\delta)\frac{\partial}{\partial n}\hat{H}(p,\gamma;q,\delta)c(q)\sin\delta d\delta dq\right]=\frac{\partial}{\partial n}\chi(p,\gamma) \tag{30}$$

IV. NUMERICAL PROBLEM

Making use of integral equation (20) and its kernel (24), the submerged foils of various geometry can be treated. However, we shall consider here foil of an elliptic strip having a constant submergence d (see Figure 2). For the present problem, we transform the variable by

$$\left.\begin{array}{l} x = c\cos\gamma,\ y = a\sin\alpha,\ z = b\cos\alpha + d, \\ \xi = c\cos\delta,\ \eta = a\sin\beta,\ \zeta = b\cos\beta + d, \end{array}\right\} \quad (31)$$

where b and d are negative and for a flat lifting surface $b = 0$, we find

$$\left.\begin{array}{l} p = \int_0^\alpha \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}\ d\alpha = aE(e,\alpha), \\ q = aE(e,\beta)\ [E;\ \text{elliptic integral of second kind}], \\ \text{and}\quad n_y = \dfrac{b\sin\alpha}{T_\alpha}, \qquad n_z = \dfrac{a\cos\alpha}{T_\alpha}, \\ n'_\eta = \dfrac{b\sin\beta}{T_\beta}, \qquad n'_\zeta = \dfrac{a\cos\beta}{T_\beta}, \end{array}\right\} \quad (32)$$

where $e^2 = \dfrac{a^2 - b^2}{a^2}$, the eccentricity of the ellipse, and

$T_\alpha = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}$, and $T_\beta = \sqrt{a^2\cos^2\beta + b^2\sin^2\beta}$.

For convenience, introducing the following notations

$$\left.\begin{array}{l} \sin\langle\alpha-\beta\rangle = \sin\alpha - \sin\beta, \quad \cos\langle\alpha\pm\beta\rangle = \cos\alpha \pm \cos\beta, \\ r(\alpha,\beta) = \sqrt{a^2\sin^2\langle\alpha-\beta\rangle + b^2\cos^2\langle\alpha-\beta\rangle}, \quad \text{and} \\ R(\alpha,\beta) = \sqrt{r^2(\alpha,\beta) + c^2\cos^2\langle\gamma-\delta\rangle}, \end{array}\right\} \quad (33)$$

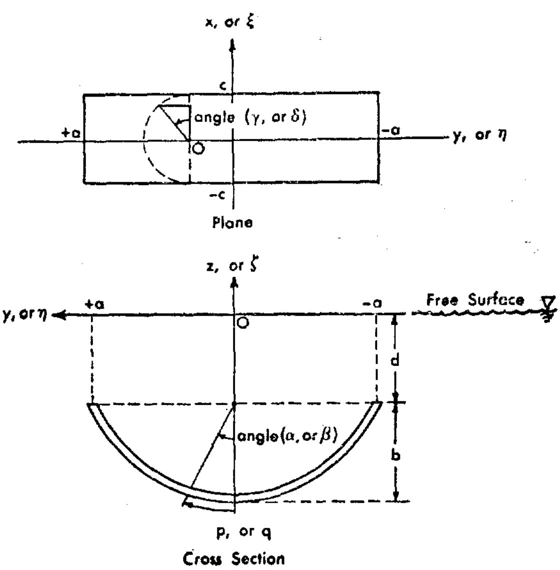

Figure 2. Submerged Foil of an Elliptic Cross Section

now for example $\frac{\partial}{\partial n}\hat{\phi}_0$ in Eq. (30) can be expressed in the form

$$\frac{\partial}{\partial n}\hat{\phi}_0(p,\gamma;q,\delta)=\frac{\partial}{\partial n}\bar{\phi}_0(\alpha,\gamma;\beta,\delta)$$

$$=\frac{a^2\cos\alpha\cos\beta}{T_\alpha T_\beta U}\left\{\frac{a^2\sin^2\langle\alpha-\beta\rangle-b^2\cos^2\langle\alpha-\beta\rangle}{r^4(\alpha,\beta)}\left[\frac{c\cos\langle\gamma-\delta\rangle}{R(\alpha,\beta)}-1\right]\right.$$

$$\left.-\frac{b^2c\cos^2\langle\alpha-\beta\rangle\cos\langle\gamma-\delta\rangle}{r^2(\alpha,\beta)R^3(\alpha,\beta)}\right\}$$

$$-ab\frac{\sin(\alpha+\beta)}{T_\alpha T_\beta U}\left\{\frac{2ab\sin\langle\alpha-\beta\rangle\cos\langle\alpha-\beta\rangle}{r^4(\alpha,\beta)}\left[\frac{c\cos\langle\gamma-\delta\rangle}{R(\alpha,\beta)}-1\right]\right.$$

$$\left.+\frac{abc\sin\langle\alpha-\beta\rangle\cos\langle\alpha-\beta\rangle\cos\langle\gamma-\delta\rangle}{r^2(a,\beta)R^3(\alpha,\beta)}\right\}$$

$$-b^2\frac{\sin\alpha\sin\beta}{T_\alpha T_\beta U}\left\{\frac{a^2\sin^2\langle\alpha-\beta\rangle-b^2\cos^2\langle\alpha-\beta\rangle}{r^4(\alpha,\beta)}\left[\frac{c\cos\langle r-\delta\rangle}{R(\alpha,\beta)}-1\right]\right.$$

$$\left.+\frac{a^2c\sin^2\langle\alpha-\beta\rangle\cos\langle\gamma-\delta\rangle}{r^2(\alpha,\beta)R^3(\alpha,\beta)}\right\} \quad (34)$$

Thus, the linear equations (30) can be written in terms of new sets of variables as

$$\frac{cU}{4\pi}\sum_{n=0}^{N}\sum_{m=0}^{M}a_{nm}\left[I_s(\alpha,\gamma;\beta,\delta)+\left(\int_{-\frac{\pi}{2}}^{\alpha-\frac{\pi}{20}}+\int_{\alpha+\frac{\pi}{20}}^{\frac{\pi}{2}}\right)\int_0^\pi\bar{g}_m(\beta)l_n(\delta)\frac{\partial}{\partial n}\bar{\phi}_0(\alpha,\gamma;\beta,\delta)\sin\delta d\delta T_\beta d\beta\right.$$

$$\left.+\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^\pi\bar{g}_m(\beta)l_n(\delta)\frac{\partial}{\partial n}\bar{H}(\alpha,\gamma;\beta,\delta)\sin\delta T_\beta d\beta\right]=\frac{\partial}{\partial n}x(\alpha,\gamma), \quad (35)$$

where

$$\bar{g}_m(\beta)=\left[\frac{E(e,\beta)}{E\left(e,\frac{\pi}{2}\right)}\right]^m\sqrt{1-\left[\frac{E(e,\beta)}{E\left(e,\frac{\pi}{2}\right)}\right]^2}$$

and the width of the neighborhood region 2ε is chosen as

$$2a\int_{\alpha-\frac{\pi}{20}}^{\alpha+\frac{\pi}{20}}\sqrt{1-e^2\sin^2\alpha}\,d\alpha.$$

I_s denotes the singular quadrature in q, and two-part quadrature involving the integrand of a finite discontinuity at $\gamma=\delta$, and that is

$$I_s=\int_{p-\varepsilon}^{p+\varepsilon}\frac{g_m(q)}{(p-q)^2}\left(\int_0^\gamma+\int_\gamma^\pi\right)l_n(\delta)[1-\mathrm{sign}(\delta-\gamma)]\sin\delta\,d\delta dq$$

$$=\int_{p-\varepsilon}^{p+\varepsilon}\frac{J(q)}{(p-q)^2}dq=\int_{\alpha-\frac{\pi}{20}}^{\alpha+\frac{\pi}{20}}\frac{\bar{J}(\beta)d\beta}{[E(e,\alpha)-E(e,\beta)]^2} \quad (36)$$

where

$$\bar{J}(\beta)=\frac{1}{a^2}\bar{g}_m(\beta)\left(\int_0^\gamma+\int_\gamma^\pi\right)l_n(\delta)[1-\mathrm{sign}(\delta-\gamma)]\sin\delta d\delta T_\beta$$

According to Watkins, et al [8], the integral I_s may be approximated by a Gaussian formular of the form

$$I_s \approx \frac{1}{5\pi}\left\{13\left[\bar{J}\left(\alpha-\frac{\pi}{20}\right)+\bar{J}\left(\alpha+\frac{\pi}{20}\right)\right]+72\left[\bar{J}\left(\alpha-\frac{\pi}{30}\right)+\bar{J}\left(\alpha+\frac{\pi}{30}\right)\right]\right.$$
$$\left.+495\left[\bar{J}\left(\alpha-\frac{\pi}{60}\right)+\bar{J}\left(\alpha+\frac{\pi}{60}\right)\right]-1360\,\bar{J}(\alpha)\right\} \qquad (37)$$

Next, we shall proceed to consider the right hand side of Eq. (35), which is the direction cosine of the exterior normal to the x-axis. When the foil $z=F(x,y)$ has the angle of attack $\tilde{\alpha}$, and the dihedral angle $\tilde{\beta}$, the slope of the foil surface in the freestream direction and that in the spanwise direction are given by $F_x=\tan\tilde{\alpha}$, and $F_y=\tan\tilde{\beta}$, respectively. Therefore, the direction cosine $\frac{\partial}{\partial n}x=\cos(n,x)$ can be expressed as

$$\frac{\partial x}{\partial n}=\frac{\tan\tilde{\alpha}}{\sqrt{1+\tan^2\tilde{\alpha}+\tan^2\tilde{\beta}}}\approx\tilde{\alpha}\cos\tilde{\beta},\ \text{for}\ \tan\tilde{\alpha}\ll 1. \qquad (38)$$

On the other hand the dihedral angle $\tilde{\beta}$ is related to a coordinate angle $\tilde{\alpha}$ by $\cos\tilde{\beta}=\cos(n,z)=a\cos\alpha/T_\alpha$, and hence the expression (38) becomes

$$\frac{\partial x}{\partial n}\approx\tilde{\alpha}\,n_z=\tilde{\alpha}\,\frac{a\cos\alpha}{T_\alpha} \qquad (39)$$

Thus, by help of Eqs. (37) and (39) we can determine the coefficient a_{nm} from the linear equations (35).

When the values of a_{nm} are known, by (21) and (28) the lift L, induced drag D and the moment at the quarter-chord M can be evaluated from the following formulae:

$$\left.\begin{aligned}
L&=-\rho\iint_S f(x,y,z)\frac{\partial z}{\partial n}dS=ac\rho U^2\sum_{n=0}^{N}\sum_{m=0}^{M}a_{nm}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\bar{g}_m(\alpha)\cos\alpha\,d\alpha\\
&\quad\times\int_0^\pi\bar{l}_n(\gamma)\sin\gamma\,d\gamma,\\
D&=-\rho\iint_S f(x,y,z)\frac{\partial x}{\partial n}dS\approx-\tilde{\alpha}\rho\iint_S f(x,y,z)\frac{\partial z}{\partial n}dS=-\tilde{\alpha}L,\\
M&=-\rho\iint_S f(x,y,z)\left[z\frac{\partial x}{\partial n}-\left(x-\frac{c}{2}\right)\frac{\partial z}{\partial n}\right]dS\\
&\approx ac^2\rho U^2\sum_{n=0}^{N}\sum_{m=0}^{M}a_{nm}\left[\tilde{\alpha}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\bar{g}_m(\alpha)\left(\frac{b}{c}\cos\alpha+\frac{d}{c}\right)\cos\alpha\,d\alpha\right.\\
&\quad\left.\times\int_0^\pi\bar{l}_n(\gamma)\sin\gamma\,d\gamma-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\bar{g}_m(\alpha)\cos\alpha\,d\alpha\int_0^\pi\bar{l}_n(\gamma)(\cos\gamma-{}^1/_2)\sin\gamma\,d\gamma\right]
\end{aligned}\right\} \qquad (40)$$

Furthermore, the lift and moment coefficient are given by

$$C_L=\frac{2L}{\rho U^2S}=\frac{c}{2a}\frac{\mathcal{R}}{E^2\left(e,\frac{\pi}{2}\right)}\sum_{n=0}^{N}\sum_{m=0}^{M}a_{nm}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\bar{g}_m(\alpha)\cos\alpha\,d\alpha\int_0^\pi\bar{l}_n(\gamma)\sin\gamma\,d\gamma,$$

and

$$C_M=\frac{2M}{\rho U^2Sc}=\frac{c}{2a}\frac{\mathcal{R}}{E^2\left(e,\frac{\pi}{2}\right)}\sum_{n=0}^{N}\sum_{m=0}^{M}a_{nm}\left[\tilde{\alpha}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\bar{g}_m(\alpha)\left(\frac{b}{c}\cos\alpha+\frac{d}{c}\right)\right. \qquad (41)$$

$$\times \cos\alpha \, d\alpha \int_0^{\pi} \bar{l}_n(\gamma) \sin\gamma \, d\gamma - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \bar{g}_m(\alpha) \cos\alpha \, d\alpha \int_0^{\pi} \bar{l}_n(\gamma)(\cos\gamma - {}^1/_2) \sin\gamma \, d\gamma \Bigg] ,$$

where $\mathcal{R}$ is the aspect ratio which is equal to $\left[2aE\left(e, \frac{\pi}{2}\right)\right]^2 / S$, S being the surface area of the lifting surface $2c \times 2aE\left(e, \frac{\pi}{2}\right)$.

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