A NOTE ON THE COMPLETENESS OF UNIFORM SPACES

By Youngshik Chang

§1 Introduction

It is introduced in [1, p. 193] that a uniform space satisfying the first axiom of countability would be complete if every Cauchy sequence converged to a point of the space, but this suspicion is unfounded.

It is natural to consider whether the above suspicion holds if the first axiom of countability is replaced with the second axiom of countability.

The main purpose of this note is to show that the answer in general is in the negative.

In §2 we will give a counterexample for the purpose.

Terminology will be adopted mainly according to [1].

§2. Lemmas and main theorem

Let X be the set of all rational numbers of the closed unit interval $[0, 1]$.

For each monotone sequences $S=\{x_i \mid i \in \omega\}$ in X, we define a subset $V(S)$ of $X\times X$ such that

$$V(S)=(X-\bigcup_i\{x_i\})\times(X-\bigcup_i\{x_i\})\cup\sum_i(x_i, x_i).$$

(LEMMA 1) *The family $\mathfrak{w}=\{V(S) \mid S$ is a monotone sequence in $X\}$ is a subbase for some uniformity $\mathscr{U}$ for X.*

(proof) It is clear.

Then we have

(LEMMA 2) *The uniform space $(X, \mathscr{U})$ has the discrete uniform topology.*

(proof) It is clear.

(LEMMA 3) *A sequence $S=\{x_i \mid i \in\omega\}$ in $(X, \mathscr{U})$ is a Cauchy sequence iff there is a $k\in\omega$ such that $x_m=x_n$ for every $m, n\geqq k$.*

(proof) It is clear.

(LEMMA 4) *Every Cauchy sequence in $(X, \mathscr{U})$ converges to one point of the space.*

(proof) It is clear by Lemma 3.

(LEMMA 5) *There exists some Cauchy net in* $(X, \mathscr{U})$ *which can not converge to a point of the space.*

(proof) For each monotone sequence S in X, let us take a subset $X(S) = X - \bigcup_i \{x_i | x_i \in S\}$ of X. It is clear that $X(S)$ is a non empty subset of X, and the family $F = \{X(S) | \ S \text{ is a monotone sequence in } X\}$ has the finite intersection property.

Let $\mathfrak{A}$ be the family of all finite intersection of members of F. $\mathfrak{A}$ is directed by $\subset$. Since each member Y_α in $\mathfrak{A}$ is a non empty subset of X we may choose a point y_α in Y_α. Then the net $\{(y_\alpha, y_\beta) | (Y_\alpha, Y_\beta) \in \mathfrak{A} \times \mathfrak{A}\}$ is eventually in each member of the family $\mathfrak{W}$ which is the subbase for the uniformity $\mathscr{U}$. Because, for an arbitrary member $V(S)$ in $\mathfrak{W}$, there is a member $X(S)$ in $\mathfrak{A}$ and (y_α, y_β) belongs to $V(S)$ whenever Y_α, Y_β follow $X(S)$. Since the net $\{(y_\alpha, y_\beta) | (Y_\alpha, Y_\beta) \in \mathfrak{A} \times \mathfrak{A}\}$ is eventually in each member of the family $\mathfrak{W}$, the net $\{y_\alpha | \ Y_\alpha \in \mathfrak{A}\}$ is a Cauchy net in $(X, \mathscr{U})$.

It is sufficient to show that the Cauchy net $\{y_\alpha | \ Y_\alpha \in \mathfrak{A}\}$ can not Converge to a point of the space X.

For an arbitrary member Y_α in $\mathfrak{A}$ there is a point y_α in Y_α which is a member of the Cauchy net $\{y_\alpha | Y_\alpha \in \mathfrak{A}\}$. Let $Y_\beta = Y_\alpha - y_\alpha$. Then Y_β is a member of $\mathfrak{A}$ and follows Y_α. Therefore there is a member y_β in Y_β, and $y_\beta \neq y_\alpha$. This shows that for every point x of X the Cauchy net is not eventually in $\{x\}$. Since $(X, \mathscr{U})$ is a discrete space the Cauchy net can not converge to a point of the space X.

We, now, have the following result by the above lemmas.

(MAIN THEOERM)

For the uniform space $(X, \mathscr{U})$ *which is constructed as above,*

(1) *the second axiom of countability is satisfied,*

(2) *every Cauchy sequence in* $(X, \mathscr{U})$ *converges to one point of the space* X, *and*

(3) *there is some Cauchy net in* $(X, \mathscr{U})$ *which does not converge to a point of the space.* (*q. e. d*)

Department of Mathematics
Dan Kook College.

REFERENCE

[1] J. L. Kelley, *General topology* (1961).