A REMARK ON NEWMAN ALGEBRAS

By F. M. Sioson

In several papers [3], [4], [5], the author has exhibited a number of independent axiom-systems for Newman and Boolean algebras that consist purely of natural identities or equational laws. As such, they have been called *equational bases.*

For any equation E involving possibly the binary operations $\cdot$, $+$ and the unary operation $-$, denote by E^{+} $(E^{\cdot})$ the equation obtained by commuting all additions (multiplications) occurring in E. Similarly, let $\bar{E}$ be the equation resulting from E through an interchange of their multiplications and additions. For instance, if E is $x+y\bar{y}=x$, then E^{+} is $y\bar{y}+x=x$ and $\bar{E}$ is $x(y+\bar{y})=x$. It is easily verified that the equational transformations $\cdot$, $+$, and $-$ generate an abelian group G_8 with eight elements which contain the four group G_4 generated by $\cdot$ and $+$. It was also shown in [5] that if $P_1, P_2, \cdots, P_n$ is any independent axiom-system of Newman (respectively Boolean) algebras, then $P_1^t, P_2^t, \cdots, P_n^t$ is also an independent axiom-system of Newman (Boolean) algebras for each $t\theta G_4 (t\theta G_8)$.

Using the results of these previous communications, we can obtain the following sharper conclusion:

THEOREM. *The only equational bases of Newman algebras out of the following pool of twelve equations:*

$N_1 : x(y+z)=xy+xz;$

$N_2 : x(y+\bar{y})=x;$ $\bar{N}_2 : x+y\bar{y}=x;$

$N_2^{\cdot} : (y+\bar{y})x=x;$

$\bar{N}_2^{+} : y\bar{y}+x=x;$

$N_3 : xy=yx;$ $\bar{N}_3 : x+y=y+x;$

$N_4 : x(y\bar{y})=y\bar{y};$

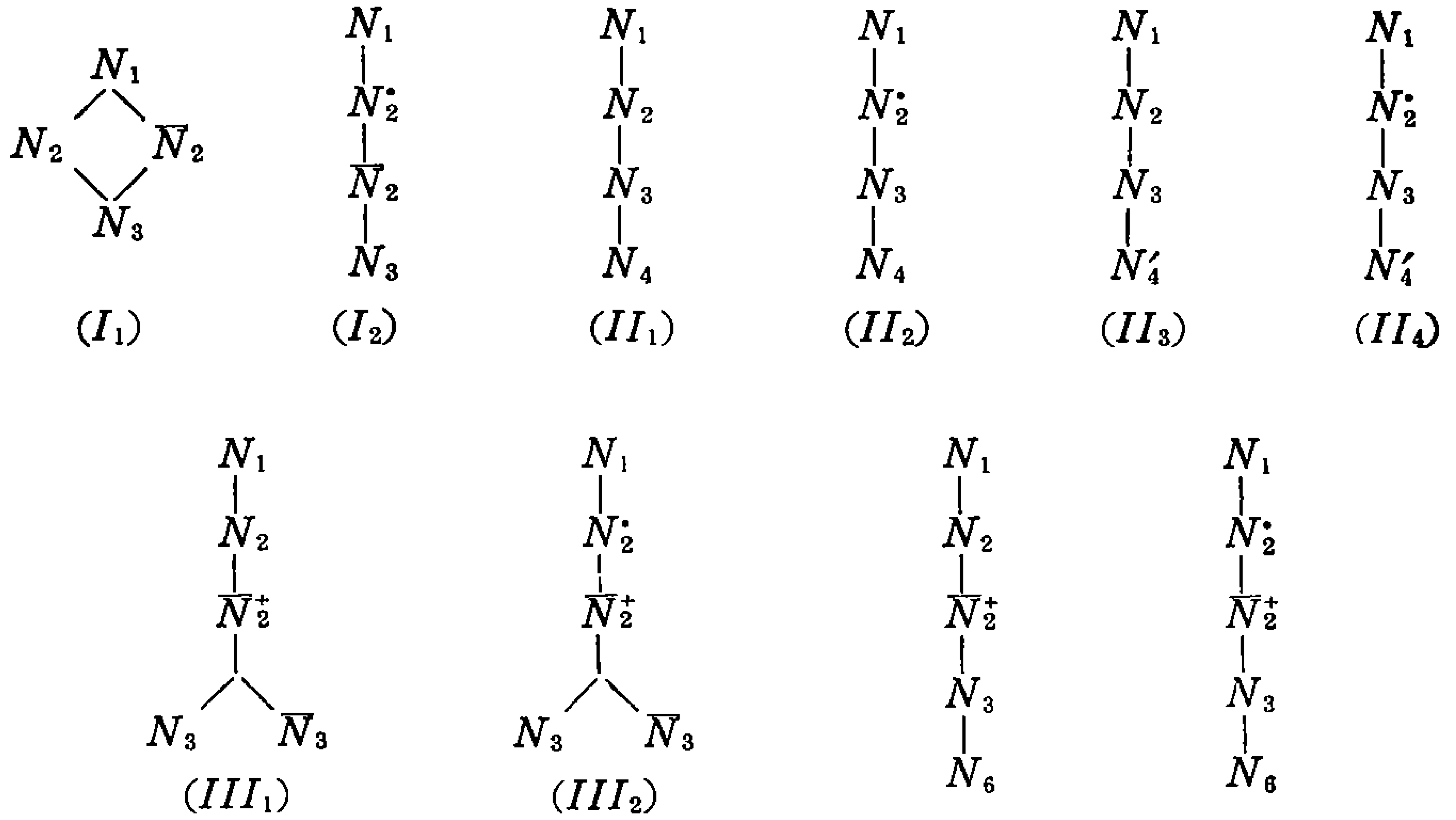
$N_4' : (y\bar{y})x=y\bar{y};$

$N_5 : xx=x;$

$N_6 : \bar{\bar{x}}=x;$

$\bar{N}_7 : x+(y+z)=(x+y)+z;$

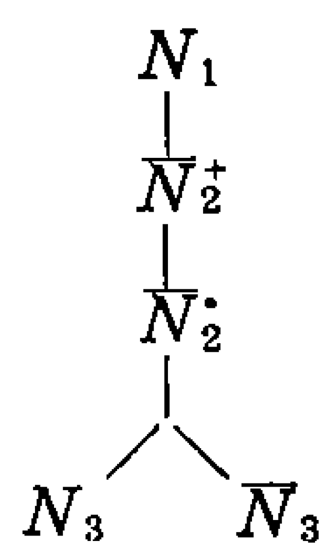
are



(I_1) (I_2) (II_1) (II_2) (II_3) (II_4) (III_1) (III_2) (IV_1) (IV_2)

and their transforms under the members of the group G_4. If to each of these equation-systems the equation $\bar{N}_5$: $x+x=x$ is added and any member t of G_8 is applied, then an equational basis of Boolean algebras arise.

Proof. The fact that I_1, II_1 are equational bases of Newman algebras has been demonstrated in article [5], while equation-system IV_1 has been shown to be a Newman equational basis in [4]. With the presence of the commutative law N_3, it is clear that I_1 and I_2, II_1, II_2, II_3, and II_4, IV_1 and IV_2 are equivalent axiom-systems. In [6], Yuki Wooyenaka also showed that the system



is an equational basis of Newman algebras. Under the presence of the commutative laws N_3 and $\bar{N}_3$, then III_1 and III_2 are each equivalent to the above Wooyenaka's basis. Thus to show that each of the systems I_2, II_2, II_3, II_4, III_1, III_2, and IV_2 forms an equational basis of Newman algebras, it will suffice to show their independence.

The independence of the system I_2 is shown by the following models:

	$0+0$	$0+1$	$1+0$	$1+1$	$0\cdot 0$	$0\cdot 1$	$1\cdot 0$	$1\cdot 1$	$\bar{0}$	$\bar{1}$
I_2N_1	0	0	0	1	0	1	1	0	1	0
I_2N_2'	0	0	1	1	0	0	0	1	1	0
$I_2\bar{N}_2$	0	1	1	1	0	0	0	1	1	1
I_2N_3	0	0	1	1	0	1	0	1	0	1

Note that for $I_2N_1 : 1(1+0)\neq 11+10$; for $I_2N_2' : (0+\bar{0})1\neq 1$; for $I_2\bar{N}_2 : 0+1\bar{1}\neq 0$; and for $I_2N_3 : 10\neq 01$.

The independence-models II_2N_2' and II_2N_3 necessary to prove the independence of N_2' and N_3 from the rest of II_2 are the same as those of I_2N_2' and I_2N_3 respectively. II_2N_1 is given by

$+$	0	1	a
0	0	1	1
1	1	1	1
a	1	1	a

$\cdot$	0	1	a
0	0	0	0
1	0	1	a
a	0	a	0

y	$\bar{y}$
0	1
1	0
a	0

and II_2N_4 by

$+$	0	1
0	0	1
1	0	1

$\cdot$	0	1
0	0	1
1	1	1

y	$\bar{y}$
0	0
1	0

Here observe that $a(a+1)\neq aa+a1$ in the first model and $1(0\bar{0})\neq 0\bar{0}$ in the second model.

The independence-models II_3N_1, II_3N_2, II_3N_4' are exactly the same as II_2N_1, I_2N_2', II_2N_4 respectively. The remaining required model II_3N_3 is given by the following:

+	0	1	a	b
0	0	1	a	b
1	1	1	1	1
a	a	1	1	1
b	b	1	1	1

•	0	1	a	b
0	0	0	0	0
1	0	1	a	b
a	0	a	a	0
b	0	b	0	b

y	$\bar{y}$
0	1
1	0
a	b
b	a

In this previous case, we have $b1 \neq 1b$ (see [3]).

For II_4, the independence-models are precisely the same as those of II_3 with the exception of II_4N_3 which is given by the following model used in [3]:

+	0	1	a	b	c	d
0	0	1	a	b	c	d
1	1	1	1	1	1	1
a	a	1	a	1	1	a
b	b	1	1	b	b	1
c	c	1	1	c	c	1
d	d	1	d	1	1	d

•	0	1	a	b	c	d
0	0	0	0	0	0	0
1	0	1	a	b	c	d
a	0	a	a	0	0	a
b	0	b	0	b	b	0
c	0	c	0	c	c	0
d	0	d	d	0	0	d

y	0	1	a	b	c	d
$\bar{y}$	1	0	b	a	d	c

Note here that $ad \neq da$. The verification of N_1 is found in [3].

For III_1N_1 we take I_2N_1. III_1N_2 we have

+	0	1
0	0	1
1	1	1

•	0	1
0	0	0
0	0	0

y	$\bar{y}$
0	1
1	0

Note that $1(y+\bar{y}) \neq 1$. For $III_1N_2^{\dagger}$ consider the collection of all finite unions of open finite, semi-infinite, and infinite intervals on the real line under the operations of taking unions (denoted by •), intersections (denoted by +) and the complement of the closure of a set (denoted by —). Then note that

$$(2,4)\overline{(2,4)}+(1,3)=((2,4)\cup\overline{(2,4)})\cap(1,3)\neq(1,3).$$

For III_1N_3 we take Y. Wooyenaka's model $\bar{P}$ found on page 86 of her paper [6]. III_1N_3 is given by the following model:

+	0	1
0	0	1
1	0	1

·	0	1
0	0	1
1	1	1

y	$\bar{y}$
0	0
1	0

In this instance, we have $1+0\neq 0+1$.

The same models used in proving the independence of III_1 may be used in proving the independence of III_2.

It remains to consider the independence of IV_2. For IV_2N_1 the model I_2N_1 suffices. $IV_2N_2^+$ may be taken as $III_1N_2^+$, while $IV_2N_2^{\cdot}$ may be taken as its dual, that is to say, the same collection of finite unions of open finite, semi-infinite, and infinite real intervals under union (this time denoted by +), intersection (this time denoted by ·) and the complement of the closure of a set (denoted again by —). In this instance,

$$((2,4)+\overline{(2,4)})(1,3)=((2,4)\cup\overline{(2,4)})\cap(1,3)\neq(1,3).$$

For IV_2N_3 we again choose II_4N_3. Finally, IV_2N is given by the following independence-model:

+	0	1
0	0	1
1	0	1

·	0	1
0	0	1
1	1	1

y	$\bar{y}$
0	0
1	0

Here $\bar{\bar{1}}\neq 1$.

After having shown that the above ten equation-systems are indeed equational bases of Newman algebras, we are now ready to prove that these are the only possible ones. It is easily seen by direct verification that the following model satisfies all the twelve equations enumerated above except N_1 since $a(b+\bar{b})\neq ab+a\bar{b}$.

+	0	1	a	b
0	0	1	a	b
1	1	1	1	1
a	a	1	a	1
b	b	1	1	b

·	0	1	a	b
0	0	0	0	0
1	0	1	0	1
a	0	a	a	0
b	0	b	0	b

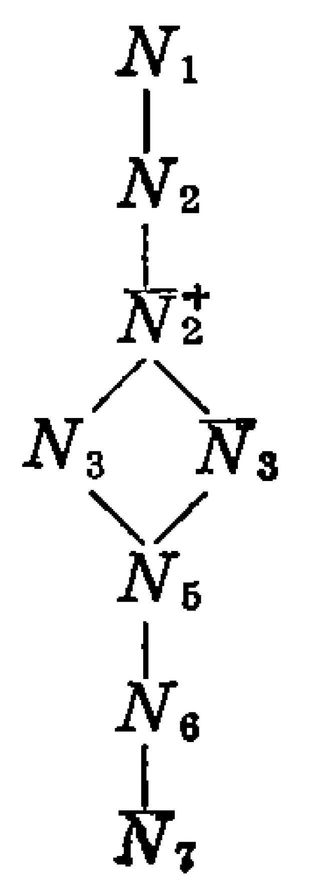
y	$\bar{y}$
0	1
1	0
a	b
b	a

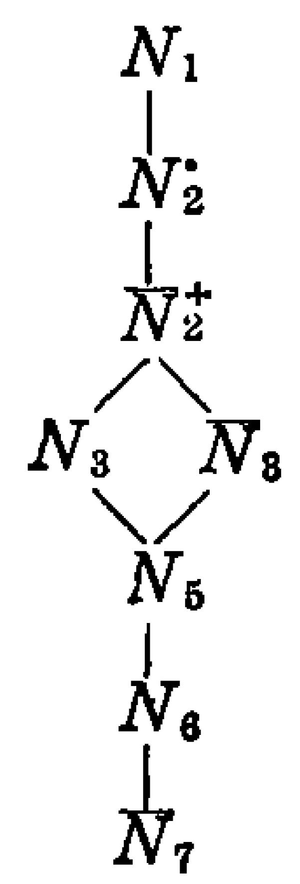
This means that N_1 is independent of the rest of the dozen equations listed in the beginning.

By exactly the same type of argument, N_3 is independent of the rest of the twelve equations by virtue of the model $\bar{P}$ of Y. Wooyenaka (page 86, [6]) which we took for III_1N_3.

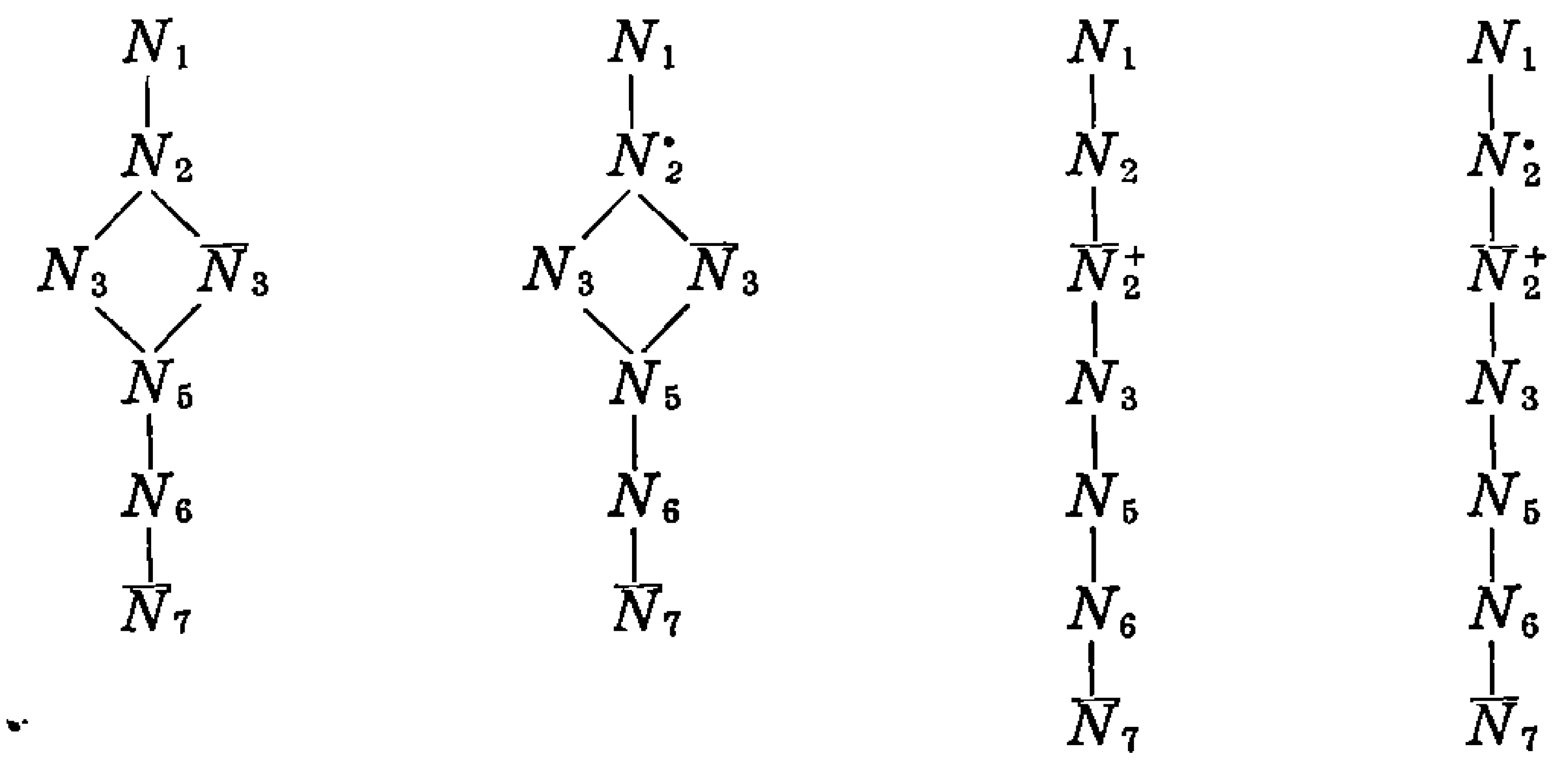
Thus, every equational basis of Newman algebras based on the pool of twelve equations given in the beginning must necessarily include N_1 and N_3. On the other hand, any such basis cannot contain both N_2 and N_2', for then it will be a dependent system. Without N_2' (N_2), equation N_2 (N_2') would then be independent of the rest of the remaining eleven equations. This is demonstrated by the model IV_2N_2'.

This implies then that every equational basis of Newman algebra based on the given pool must contain all three N_1, N_2 and N_3 or N_1, N_2', N_3. The equational bases given above are examples of such bases. In order to seek other possible equational bases for Newman algebras, the equations $\bar{N}_2$, N_4, and N_4' should be excluded from our modified pool, thus reducing the original set of twelve to either one of the following:



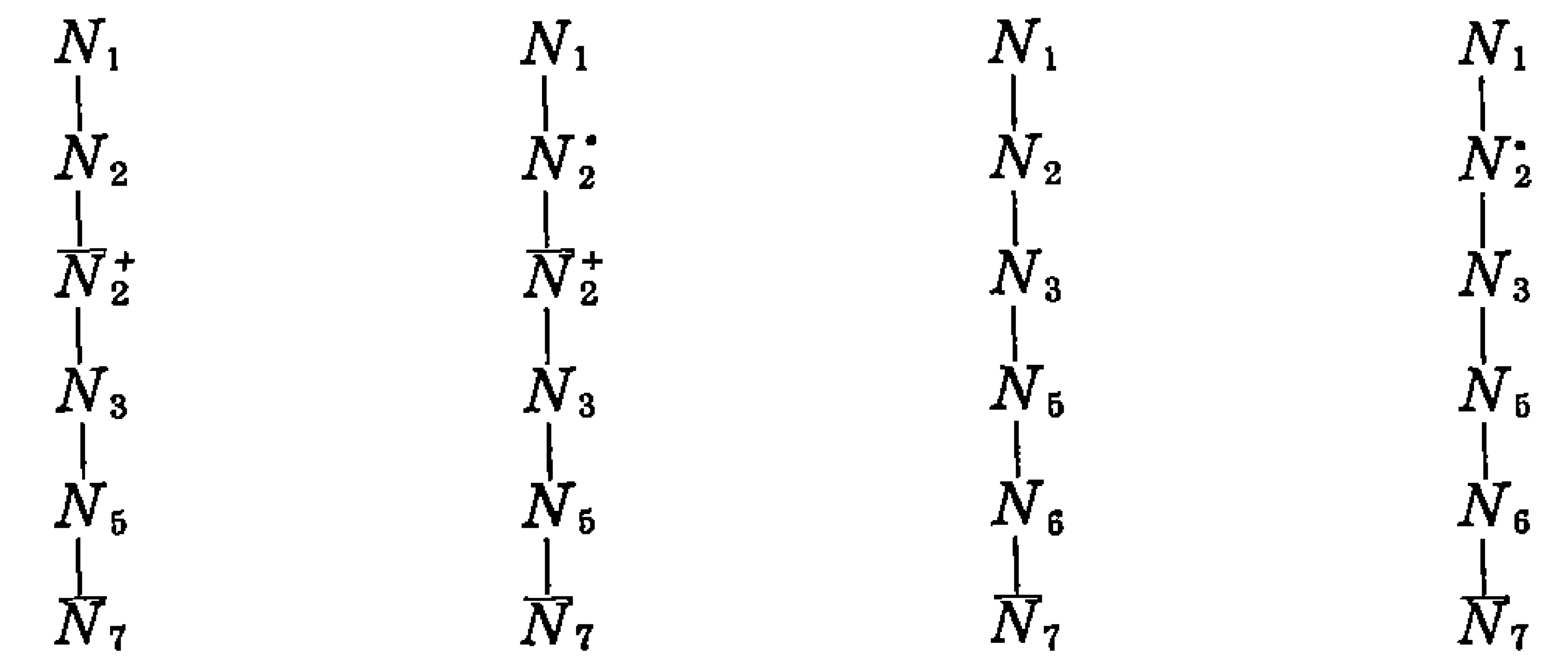


Since each of these properly contains an equational basis of Newman algebras, there cannot then be a Newman equational basis out of the original twelve with eight axioms or equations. The possible equational bases with seven equations must also exclude $\overline{N}_2^+$ or $\overline{N}_3$ (since III_1—III_2 and IV_1—IV_2 are already known to be equational bases). The following are then the only possible candidates for Newman equational bases with seven axioms:



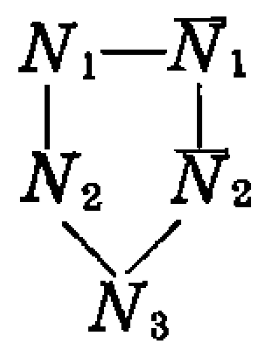
The third and fourth systems contains respectively IV_1 and IV_2 and hence are dependent, while the first and second are incomplete, since $\overline{N}_2$ is independent of them. This is shown by the model $III_1\overline{N}_2^+$ used in the proving the independence of $\overline{N}_2^+$ from the rest of system III_1.

From the third and fourth systems, the equation-systems with six axioms that may possibly be bases are



The last two of these systems are incomplete being subsets of incomplete systems of equations for Newman algebras. The first two are also incomplete, since $\overline{N}_3$ or N_6 cannot be derived from them. The independence of $\overline{N}_3$ or N_6 is effected by model IV_2N_6.

From [2], it is implicit that our original pool of 12 equations N_1, N_2, $N_2^{\cdot}$, $\overline{N}_2$, $\overline{N}_2^+$, N_3, $\overline{N}_3$, N_4, N_4', N_5, N_6, and $\overline{N}_7$ hold in any Newman algebra. Moreover, any Newman algebra satisfying the equation $\overline{N}_5 : x+x=x$ is a Boolean algebra. To show this, it is sufficient to derive $N_1 : x+yz=(x+y)(x+z)$ from the axioms of Newman algebra with $\bar{N}_5$ adjoined and hence derivet the equational basis



of Boolean algebras (see [3]). This is easily effected as follows:

(a) $x+xy=x$.

$x=x(y+\bar{y})=xy+x\bar{y}=(xy+xy)+x\bar{y}=xy+(xy+x\bar{y})=xy+x(y+\bar{y})$
$=xy+x=x+xy \qquad (N_2,\ N_1, \bar{N}_5, \bar{N}_7, N_1, N_2, \bar{N}_3)$.

(b) $x+yz=(x+y)(x+z)$.

$(x+y)(x+z)=(x+y)x+(x+y)z=x(x+y)+z(x+y)$
$=(xx+xy)+(zx+zy)=(x+xy)+(xz+yz)=x+(xz+yz)=(x+xz)+yz$
$=x+yz \qquad (N_1, N_3-N_3, N_1-N_1, N_5-N_3-N_3, \text{(a)}, \bar{N}_7, \text{(a)})$.

The independence of each of the equation-systems I_1, I_2, II_1, II_2, II_3, II_4, III_1, III_2, IV_1, IV_2 with $\bar{N}_5$ adjoined follows from the observation that each of the models used in proving the independence of I_1, I_2, II_1, II_2, II_3, II_4, III_1, III_2, IV_1, IV_2 all satisfy the equation $\bar{N}_5 : x+x=x$ and that the equation $\bar{N}_5$ itself in independent of Newman algebras as shown by the following example of a two element Newman algebra:

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

y	$\bar{y}$
0	1
1	0

Here $1+1 \neq 1$.

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