A NOTE ON ADJOINT KAEHLERIAN MANIFOLD

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Introduction.

In this note, we investigate the few properties of the sectional curvatures of the adjoint Kaelerian manifold, which is defined in the previous paper [1].

Let us consider a $(2n+1)$-dimensional differentiable manifold X_{2n+1} of class C^{∞}, which is covered by the real coordinate neighborhood system (x^{i}), where i, j, k is taken by the indices $1, 2, \cdots, n; \bar{1}, \bar{2}, \cdots, \bar{n}; \infty$. We put

(1) $$z^{\alpha}=x^{\alpha}+ix^{\bar{\alpha}},\quad z^{\bar{\alpha}}=x^{\alpha}-ix^{\bar{\alpha}},\quad z^{\infty}=x^{\infty},$$

where α, β, γ and $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ are taken by the indices $1, 2, \cdots, n$ and $\bar{1}, \bar{2}, \cdots, \bar{n}$ respectively. Then we know that $(x^{\alpha}, x^{\bar{\alpha}}, x^{\infty})$ assigns to $(z^{\alpha}, z^{\bar{\alpha}}, z^{\infty})$. and conversely.

If it is possible to choose a coordinate neighborhood system, in such that, in the domain $U \cap U'$ of two neighborhoods $U(z^{i})$ and $U'(z^{i'})$, it holds

(2) $$z^{\alpha'}=z^{\alpha'}(z^{\alpha}),\quad z^{\bar{\alpha}'}=z^{\bar{\alpha}'}(z^{\bar{\alpha}}),\quad z^{\infty'}=z^{\infty'}(z^{\infty}),$$
$$\left|\frac{\partial z^{\alpha'}}{\partial z^{\alpha}}\right|\cdot\left|\frac{\partial z^{\bar{\alpha}'}}{\partial z^{\bar{\alpha}}}\right|\cdot\frac{\partial z^{\infty'}}{\partial z^{\infty}}\neq 0 .$$

We say that the manifold X_{2n+1} admits an adjoint complex structure and we call X_{2n+1} an adjoint complex manifold.

Now, we assume that our adjoint complex manifold admits a Riemannian metric

(3) $$ds^{2}=g^{ji}dz^{j}dz^{i},$$

where symmetric tensor g^{ji} is adjoint self-conjugate and satisfies

(4) $$(g^{ji})=\begin{pmatrix} 0 & g^{\beta\bar{\alpha}} & 0 \\ g^{\bar{\beta}\alpha} & 0 & 0 \\ 0 & 0 & g_{\infty\infty} \end{pmatrix}.$$

Then the metric form (3) is represented by the by the form

(5) $$ds^{2}=2g_{\beta\bar{\alpha}}dz^{\beta}dz^{\bar{\alpha}}+g_{\infty\infty}dz^{\infty}dz^{\infty}.$$

We call this metric satisfying (4) an adjoint Hermitian metric and the adjoint complex manifold with this metric an adjoint Hermitian manifold.

Next, for the Christoffel symbols on this adjoint Hermitian manifold, if they satisfy that

(6) $\Gamma^h_{ji}=0,$ excepting $\Gamma^\alpha_{\gamma\beta}$, $\Gamma^{\bar\alpha}_{\bar\gamma\bar\beta}$ and $\Gamma^\infty_{\infty\infty}$,

or, as the equivalent condition of this,

(7)
$$\begin{aligned}&\partial_\gamma g_{\beta\bar\alpha} = \partial_\beta g_{\gamma\bar\alpha}, && (\partial_{\bar\gamma} g_{\bar\beta\alpha} = \partial_{\bar\beta} g_{\bar\gamma\alpha}),\\ &\partial_\infty g_{\beta\bar\alpha} = 0, &&\\ &\partial_\tau g_{\infty\infty} = 0, && (\partial_{\bar\tau} g_{\infty\infty} = 0),\end{aligned}$$

We say that the adjoint Hermitian manifold admits an adjoint Kaehlerian condition and we call this manifold X_{2n+1} an adjoint Kaehlerian manifold.

On the adjoint Kaehlerian manifold, the Christoffel symbols are represented by

(8) $$\Gamma^\alpha_{\gamma\beta} = g^{\alpha\bar\tau}\partial_\gamma g_{\beta\bar\tau},\quad \Gamma^{\bar\alpha}_{\bar\gamma\bar\beta} = g^{\bar\alpha\tau}\partial_{\bar\gamma} g_{\bar\beta\tau},\quad \Gamma^\infty_{\infty\infty}=\partial_\infty \log\sqrt{g_{\infty\infty}},$$

and the others are zero.

And, for the curvature tensor, we can see that only the components of the form

(9) $$R_{\delta\gamma\beta}{}^{\alpha},\quad R_{\bar\delta\bar\gamma\bar\beta}{}^{\bar\alpha},\quad R_{\delta\bar\gamma\bar\beta}{}^{\alpha},\quad R_{\bar\delta\gamma\beta}{}^{\bar\alpha},$$

and

(10) $$R_{\delta\bar\gamma\beta\bar\alpha},\quad R_{\bar\delta\gamma\bar\beta\alpha},\quad R_{\bar\delta\gamma\beta\bar\alpha},\quad R_{\delta\bar\gamma\bar\beta\alpha},$$

can be different from zero. And they are represented by

(11) $$R_{\bar\delta\gamma\beta}{}^{\alpha}=\partial_{\bar\delta}\Gamma^\alpha_{\gamma\beta},\qquad R_{\delta\bar\gamma\bar\beta}{}^{\bar\alpha}=\partial_\delta\Gamma^{\bar\alpha}_{\bar\gamma\bar\beta}.$$

Note.

We now introduce a sectional curvature K determined by linearly independent vectors u^i and v^i:

(12) $$K=\frac{R_{lkji}v^l u^k v^j u^i}{(g_{kj}g_{li}-g_{lj}g_{ki})\ v^l u^k v^j u^i}.$$

If this sectional curvature K is invariant for all possible two dimensonal section, then the curvature tensor must have the form

(13) $$R_{lkji}=K(g_{kj}g_{li}-g_{lj}g_{ki}).$$

In the present case, from (4), this reduces into the equations

(14) $$R_{\delta\gamma\beta\alpha}=Kg_{\gamma\beta}g_{\delta\alpha},$$

(15) $$0=R_{\delta\bar{\gamma}\beta\alpha}=K(g_{\bar{\gamma}\beta}g_{\delta\alpha}-g_{\delta\beta}g_{\bar{\gamma}\alpha}),$$

(16) $$0=R_{\infty\gamma\beta\infty}=Kg_{\gamma\beta}g_{\infty\infty}.$$

Since, from Ricci identity, we have

$$R_{\delta\gamma\beta\alpha}=R_{\delta\alpha\beta\gamma},$$

(14) reduces into

(17) $$Kg_{\gamma\beta}g_{\delta\alpha}=Kg_{\alpha\beta}g_{\delta\gamma}.$$

Transvecting $g^{\bar{\gamma}\beta}g^{\delta\alpha}$ to (17), we have

(18) $$n(n-1)K=0.$$

And, transvecting $g^{\bar{\gamma}\beta}g^{\delta\alpha}$ to (15) and $g^{\gamma\beta}g^{\infty\infty}$ to (16), we have

(19) $$n(n-1)K=0,$$

and

(20) $$nK=0.$$

Thus, from (18), (19) and (20), we obtain

(21) $$K=0.$$

Hence, we have

THEOREM 1. *At every point of an adjoint Kaehlerian manifold, if the sectional curvature is invariant for all possible two dimensional sections, then the manifold is flat.*

We consider the vectors u^i and v^i satisfying the conditions

(22) $$v^{\alpha}=iu^{\alpha},\qquad v^{\bar{\alpha}}=-iu^{\bar{\alpha}},\qquad v^{\infty}=iu^{\infty}.$$

Then we can see that these vectors are linearly independent.

For this section $(u^i,\ v^i)$, we have

(23) $$R_{lkji}v^l u^k v^j u^i=-4R_{\delta\gamma\beta\alpha}u^{\delta}u^{\gamma}u^{\beta}u^{\alpha},$$

(24) $$(g_{kj}g_{li}-g_{lj}g_{ki})v^l u^k v^j u^i=-4g_{\bar{\gamma}\beta}g_{\delta\bar{\alpha}}u^{\delta}u^{\bar{\gamma}}u^{\beta}u^{\bar{\alpha}},$$

and consequently,

$$(25)\qquad K=\frac{R_{\delta\bar{\gamma}\beta\bar{\alpha}}\,u^{\delta}u^{\bar{\gamma}}u^{\beta}u^{\bar{\alpha}}}{g_{\bar{\gamma}\beta}g_{\delta\bar{\alpha}}u^{\delta}u^{\bar{\gamma}}u^{\beta}u^{\bar{\alpha}}}=\frac{2R_{\delta\bar{\gamma}\beta\bar{\alpha}}u^{\delta}u^{\bar{\gamma}}u^{\beta}u^{\bar{\alpha}}}{(g_{\bar{\gamma}\beta}g_{\delta\bar{\alpha}}+g_{\bar{\alpha}\beta}g_{\delta\bar{\gamma}})u^{\delta}u^{\bar{\gamma}}u^{\beta}u^{\bar{\alpha}}}.$$

If we assume that, at all point of the manifold, the sectional curvature for all the section satisfying (22) is invariant, then it holds

$$(26)\qquad R_{\delta\bar{\gamma}\beta\bar{\alpha}}=\frac{1}{2}K(g_{\bar{\gamma}\beta}g_{\delta\bar{\alpha}}+g_{\bar{\alpha}\beta}g_{\delta\bar{\gamma}}).$$

On the other hand, from the Bianchi identify:

$$\nabla_m R_{lkji}+\nabla_l R_{kmji}+\nabla_k R_{mlji}=0,$$

we have

$$(27)\qquad \nabla_{\varepsilon}R_{\delta\bar{\gamma}\beta\bar{\alpha}}+\nabla_{\delta}R_{\bar{\gamma}\varepsilon\beta\bar{\alpha}}+\nabla_{\bar{\gamma}}R_{\varepsilon\delta\beta\bar{\alpha}}=0,$$

$$\nabla_{\bar{\varepsilon}}R_{\delta\bar{\gamma}\beta\bar{\alpha}}+\Delta_{\delta}R_{\bar{\gamma}\bar{\varepsilon}\beta\bar{\alpha}}+\nabla_{\bar{\gamma}}R_{\bar{\varepsilon}\delta\beta\bar{\alpha}}=0,$$

$$\nabla_{\infty}R_{\delta\bar{\gamma}\beta\bar{\alpha}}+\nabla_{\delta}R_{\bar{\gamma}\infty\beta\bar{\alpha}}+\nabla_{\bar{\gamma}}R_{\infty\delta\beta\bar{\alpha}}=0.$$

From (10), these reduce into

$$(28)\qquad \nabla_{\varepsilon}R_{\delta\bar{\gamma}\beta\bar{\alpha}}=\nabla_{\delta}R_{\varepsilon\bar{\gamma}\beta\bar{\alpha}},$$

$$\nabla_{\bar{\varepsilon}}R_{\delta\bar{\gamma}\beta\bar{\alpha}}=\nabla_{\bar{\gamma}}R_{\delta\bar{\varepsilon}\beta\bar{\alpha}},$$

$$\nabla_{\infty}R_{\delta\bar{\gamma}\beta\bar{\alpha}}=0.$$

Substituting (26) into (28), we have

$$(29)\qquad \nabla_{\varepsilon}K(g_{\bar{\gamma}\beta}g_{\delta\bar{\alpha}}+g_{\bar{\alpha}\beta}g_{\delta\bar{\gamma}})=\nabla_{\delta}K(g_{\bar{\gamma}\beta}g_{\varepsilon\bar{\alpha}}+g_{\bar{\alpha}\beta}g_{\varepsilon\bar{\gamma}}),$$

$$\nabla_{\bar{\varepsilon}}K(g_{\bar{\varphi}\beta}g_{\delta\bar{\alpha}}+g_{\bar{\alpha}\beta}g_{\delta\bar{\gamma}})=\nabla_{\bar{\gamma}}K(g_{\bar{\varepsilon}\beta}g_{\delta\bar{\alpha}}+g_{\bar{\alpha}\beta}g_{\delta\bar{\varepsilon}}),$$

$$\nabla_{\infty}K(g_{\bar{\gamma}\beta}g_{\delta\bar{\alpha}}+g_{\bar{\alpha}\beta}g_{\delta\bar{\gamma}})=0.$$

Transvecting $g^{\bar{\gamma}\delta}g^{\delta\bar{\alpha}}$ to (29), these reduce into

$$(30)\qquad (n-1)\ \nabla_{\varepsilon}K=0,$$

$$(n-1)\ \nabla_{\bar{\varepsilon}}K=0,$$

$$\nabla_{\infty}K=0.$$

Hence, for $n>1$

(31) $$\nabla_i K=0$$

Thus, we obtain

THEOREM 2. *If, at all points of an adjoint Kaehlerain manifold, the sectional curvature for all the section satisfying* (22) *is invariant, then the curvature tensor has the form* (26) *and it is an absolute constant.*

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