QUASI-COMPACT SPACE AND TOPOLOGICAL PRODACT OF MINIMAL HAUSDORFF SPACE

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0. Introduction

In the paper [5] M. P. BRRI posed a problem *"Is the topological product of minimal Hausdorff spaces necessarily minimal Hausdorff?"*

We prove in this paper that the product of minimal Hausdorff spaces is minimal Hausdorff. For the proof of the above problem, we consider a space that has some property of compact space and we define it as a quasi-compact space and prove that arbitrary product of quasi-compact spaces is also quasi-compact. The methods of proof are used in THEOREM 1.3. are identical with those used in the proof of the product of compact spaces is compact.

1. Qasi-compact spaces.

1.1. Definition. A topological space is quasi-compact if and only if each family of open sets which has the finite intersection property has a non-void adherent set.

1.2. Definition. $\{O_\alpha\}$ is a family of subsets of topological space X, then we call $\wedge\{\overline{O}_\alpha\}$ as the adherent set of $\{O_\alpha\}$.

1.3. THEOREM. *Let $\{X_a : a \in A\}$ be a family of quasi-compact spaces, then the cartesian product $X=\Pi X_a$ is quasi-compact relative to the product topology.*

Proof. Let $\mathscr{A}$ be a family of open sets of X which has the finite intersection property. We shall prove that $\mathscr{A}$ has a non-void adherent set. The class of all open famillies which posses the finite intersection property is of finite character and consequently we may assume $\mathscr{A}$ is maximal with respect to this property by Tuke's Lemma. Because $\mathscr{A}$ is maximal with respect to this property each open set which contains a member of $\mathscr{A}$ belongs to $\mathscr{A}$ and the intersection of two members of $\mathscr{A}$ belongs to $\mathscr{A}$. Moreover, if B is an open set which intersects each member of $\mathscr{A}$, then $B \in \mathscr{A}$ by maximality. Finally, the family of projections of members of $\mathscr{A}$ into a coordinate space has the finite intersection property and is an open family, hence it is possible to choose a point x_a in $\wedge\{\overline{P_a[B]} : B\epsilon\mathscr{A}\}$. Here P_a denotes the projection into coodinate space, and $\bar{B}$ denotes the closure of B. The point x whose a-th coordinate is x_a then has the property: each open neighborhood U of x_a intersects $P_a[B]$ for every B in $\mathscr{A}$ or equivelently $P_a^{-1}[U]\epsilon\mathscr{A}$, for each open neighborhood U of x_a in X_a. Therefore finite intersections of sets of this form belongs to $\mathscr{A}$. Then each neighborhood of x which belongs to the defining base for the product topology belongs to $\mathscr{A}$ and hence intersects each member of $\mathscr{A}$. Therefore x belongs to $\bar{B}$ for each B in $\mathscr{A}$, and the theorem is proved.

1.4. THEOREM. *A quasi-compact subspace of quasi-compact T_2 space X is closed in X.*

Proof. Let M be a subset of quasi-compact T_2 space X, and let M be not closed in X. Then there is a point x which is contained in $\overline{M}$ but not contained in M. Let $\{U_\alpha : \alpha\epsilon\Delta\}$ be the family of open neighborhoods of x in X. Then $\{U_\alpha\cap M : \alpha\epsilon\Delta\}$ is an open family of the subspace M with finite intersection property. But the common part of the closures of $U_\alpha\cap M$ in the subspace M is void. Hence M is not a quasi-compact subspace. This proves our theorem.

2. Minimal Hausdorff spaces.

We shall write some definitions and theorems those are needed in the proof of Theorem 2.6., and all of these can be found in the paper [5].

2.1. Definition. A filter base Ψ on a set X is said to be weaker than a filter base φ on X, if for each $F\epsilon\Psi$, there exists some $G\epsilon\varphi$ such that $G\subseteq F$.

2.2. Definition. Given a topological space X. An open filter base on X is a filter base composed exclusively of open sets. A closed filter base on X is a filter base composed exclusively of closed sets.

2.3. Definition. A topological space $(X, \mathscr{U})$ is said to be minimal Hausdorff if $\mathscr{U}$ is Hausdorff and there exists no Hausdorff topology on X strictly weaker than $\mathscr{U}$.

In the comparison of topologies, a topology $\mathscr{U}$ will be weaker than a topology $\mathcal{T}$ if $\mathcal{T}$ is a subfamily of $\mathscr{U}$.

We shall use following characterization of minimal Hausdorff spaces as given in [1], [2] and [5].

2.4. THEOREM. *A necessary and sufficient condition that a Hausdorff space* $(X, \mathfrak{M})$ *be minimal Hausdoff is that* $\mathfrak{M}$ *satisfies property* (i) (ii): (i) *Every open filter base has an adherent point,* (ii) *if an open filter base has a unique adherent point, then it converges to this point.*

2.5. THEOREM. *A Hausdorff space which satisfies* (ii) *also satisfies* (i).

2.6. THEOREM. *Let* $\{X_a : a\epsilon\Delta\}$ *be a family of non-empty minimal Hausdorff spaces, then the product space* $X=\Pi X_a$ *is a minimal Hausdorff with the product topology.*

Proof. Let $\{O_\beta : \beta\epsilon\Delta\}$ be an open filter base such that $O_\beta=\Pi O^a_\beta$, O^a_β is an open set in X_a. Here Π denotes the cartesian product. Let us assume $\{O_\beta : \beta\epsilon\Delta\}$ has a unique adherent point d. Then $d=\wedge\{\overline{O}_\beta : \beta\epsilon\Delta\}$, and let d_a be the component of d, in X_a. It follows that $d_a=\wedge\{\overline{O}^\alpha_\beta : \beta\epsilon\Delta\}$. Hence by the minimal property of X_a, for any neighborhood V of d_a, there exists an O^a_β such that $O^a_\beta \subset V$. Hence for any neighborhood U of d, there is an $O_\beta\subset U$. Hence we proved $\{O_\beta : \beta\epsilon\Delta\}$ converges to d(*). We shall call an open set O in X a basic open set if it is the cartesian product of open sets of the factor spaces, and a closed set a basic closed set if it is the cartesian product of closed sets of the factor spaces. Now let $\{O_\beta : \beta\epsilon\Delta\}$ be an arbitrary open filter base with a unique adherent point d. For each $O_\beta\epsilon\{O_\beta : \beta\epsilon\Delta\}$, A_β be the the least basic closed set containing O_β. Then $A_\beta={}_a\Pi P_a\,[A_\beta]$, let I^a_β be the greatest open set contained in $P_a\,[A_\beta]$. Then we can see that $(\prod_a I^a_\beta)\supseteq O_\beta$ and $\overline{\prod_a I^a_\beta}=A_\beta$. Here P_a denotes the projection into factor space X_a. Now we shall show that d is the unique adherent point of $\{A_\beta : \beta\epsilon\Delta\}$. Suppose $x\neq d$ and $x\epsilon\wedge\{A_\beta\}$, then there is an open set O_β such that $O_\beta\epsilon\{O_\beta : \beta\epsilon\Delta\}$ and $x\notin\overline{O}_\beta$. Since the product space X is Hausdorff we can find an open neighborhood V of x and an open neighborhood U of d such that the intersection of U and V is void. Let U' be the complement of U, then U' contains $\overline{V}$. Without loss of generality we can assume U and V be basic open sets. Hence we can put $V=\prod_a P_a\,[V]$ and $U=\prod_a P_a[U]$. Since the intersection of U and V is void we get $d\notin\left(\overline{\prod_a P_a\,[V]}\right)$. Whence there exists a $\overline{P_b[V]}$ such that $P_b\,[d]\notin\overline{P_b[V]}$. Let M be the open set in X which satisfies $P_b[M]=P_b\,[V]$ and $P_a\,[M]=X_a$ if $a\neq b$. Then $(A_\beta-M)$ is a closed set for each $\beta\epsilon\Delta$. If for some β, O_β and M have a void intersection then.

$$(A_\beta - M) = \prod_{a \neq b} P_a[A_\beta] \times (P_b[A_\beta] - P_b[V])$$

and $A_\beta - M$ must contain O_β, it contradicts the definition of A_β. Hence O_β and M can not have a void intersection, so that $\{O_\beta \cap M : \beta \epsilon \Delta\}$ has finite intersection property. Hence by the Theorem 1.3. $\{O_\beta \cap M : \beta \epsilon \Delta\}$ has an adherent point m that does not cincide with d. By construction m is contained in $\{\overline{O}_\beta : \beta \epsilon \Delta\}$, it is a contradiction. Thus we proved that $\{A_\beta : \beta \epsilon \Delta\}$ has the unique adherent point d.

It follows that the open filter base generated by $\{\prod_a I^a_\beta : \beta \epsilon \Delta\}$ has unique adherent point d, and by (*) it converges to d. Hence $\{O_\beta : \beta \epsilon \Delta\}$ converges to d. That proves our Theorem.

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