A NOTE ON SEPERATION AXIOMS WEAKER THAN T_1

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Introduction.

In the article *"seperation axioms between* T_0 *and* T_1*"* [1], C.E. Aull and W.J. Thron introduced many seperation axioms between T_0 and T_1, also studied there inclusion relations, and their behavior under a strengthening of topology. In the section 7 of the paper they posed a problem:

PROBLEM. *Is there a seperation axioms* T_α *weaker than* T_1 *such that a normal* T_α *space is* T_4.

In this paper, we provide an axiom T_α which is weaker than T_1 such that a normal T_α space is T_4. We also prove that there exists no seperation axiom T_α which is preserved under a strengthening of the topology and weak than T_1 such that a normal T_α space is T_4.

Through this paper, all the terminologies used here are same with [1], but it is denoted the point closure of x in a topological space X by $\bar{x}$ simply.

§ 1. Seperation axioms preserved under the strengthening of topology.

In this section we prove the problem in the case of the seperation axioms preserved under a strengthening of topology. We state the following theorem and prove it.

THEOREM 1. *There exists no seperation axiom* T_α *which is preserved under a strengthening of topology and weaker than* T_1 *such that a normal* T_α *space is* T_4.

To prove the above theorem, we introduce a topological space, Let X be an aggregate with more than two elements. After having fixed the elements $\{a, b\}$, let all the subsets A_α of X and $A_\alpha \cup \{a, b\}$ be $\mathcal{O}$. Then we can denote $\mathcal{O}$ as follow:

$\mathcal{O} = [A_\alpha, A_\alpha \cup \{a, b\} : A_\alpha \subset X-a,\ b \in X-a,\ \{a, b\}$ *are two fixed elements of* $X]$

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DEFINITION. When X is an aggregate, and $\mathcal{O}$ is the family of the sets constructed above, we call $\mathcal{O}$ a *N_b^a-class of X.*

Now, we prove two propositions, and then have THEOREM 1 immediately.

PROPOSITION. 1. *Let X be an aggregate, and $\mathcal{O}$ be an N_b^a-class of X. Taking $\mathcal{O}$ as the familly of closed sets of X, we have a topological space $(X, \mathcal{O})$. Then $(X, \mathcal{O})$ is a normal space which has $\{a, b\}$ as the closure of a.*

PROOF. We shall show that, for each pair of disjoint closed sets A and B, there exist two disjoint open sets U and V such that $A \subset U$ and $B \subset V$. Then we know this is proved by the following two cases;

(1) A, B are both contained in X-a.

(2) A is contained in X-a, a is contined in B, hence $\{a, b\} \subset B$.

In case (1) at least one of A, B does not contain b, hence we assume A doesnot contain b. Denote the complemented set of A by A', then A' contins B and A' has the form $A' = [A_\alpha \cup \{a, b\} : A_\alpha \subset X - a]$. Hence, A' is closed and open, therefore, A is closed and open. If we let $A = U$, $A' = V$, we have two disjoint open sets we need.

In case (2) let us denote the complement of B by B', then $B' \supset A$ and B' is contained in X-a. Hence B' is open and closed, therefore, B is open and closed. If we let $B' = U$, $B = V$ we have two disjoint open sets we need. Finally it is clear by the construction of $\mathcal{O}$ that the point closure of a is $\{a, b\}$.

DEFINITION. If $(X, \mathcal{O})$ is a topological space having $\mathcal{O}$ as the familly of closed sets, and $\mathcal{O}$ is a N_b^a-class of X, then we call $(X, \mathcal{O})$ T_N-space.

PROPOSITION 2. *Let T_α be a seperation axiom which is preserved under a strengthening of topology and weaker than T_1, and let $(X, \mathcal{L})$ be a topological space which satisfies T_α axiom, but does not satisfy T_1. Then there exists a T_N-space $(X, \mathcal{O})$ such that $\mathcal{L} \supset \mathcal{O}$, hence $(X, \mathcal{O})$ satisfies T_α.*

PROOF. Since a seperation axiom which is preserved under a strengthening of topology simply insists the existence of some closed sets in the topological space, to prove above proposition it is sufficient to show that there is a T_N-space $(X, \mathcal{O})$ which contains all of the closed sets the T_α-space $(X, \mathcal{L})$ is containing. Since $(X, \mathcal{L})$ does not satisfy T_1-axiom, there exists an element a such that the point closure of a is not a. Let $\bar{a}$ denote the point closure of a, then we can denote $\bar{a}=\{a, b\cdots\cdots\}$, $a \neq b$. We fix two elements $\{a, b\}$ and let us put $\mathcal{O}=[A_\alpha, A_\alpha \cup \{a, b\} : A\alpha \subset X\text{-}a]$. Then $\mathcal{O}$ is a N_b^a-class of X. Taking $\mathcal{O}$ as the familly of closed sets of X, we get a topological space $(X, \mathcal{O})$. We shall show if $C \in \mathcal{L}$, then $C \in \mathcal{O}$.

(1) If $C \not\ni a$, then $C \in \mathcal{O}$ is clear.

(2) If $C \ni a$ then $C \supset \{a, b\}$, and we can decompose set C as follow $C=A_\alpha \cup \{a, b\}$, where $A_\alpha \subset X\text{-}a$. By the definition of $\mathcal{O}$ we have $A_\alpha \cup \{a, b\} \in \mathcal{O}$, that is $C \in \mathcal{O}$, and thus we have proved it.

From PROPOSITIONS 1 and 2, if $(X, \mathcal{L})$ is a topological space satisfying T_α—axiom which is preserved under a strengthening of topology and weaker than T_1, and the space does not satisfy T_1, then we always can construct a T_N-space $(X, \mathcal{O})$ which is normal but not T_1 and satisfies T_α. Hence we proved THEOREM 1.

§ 2. Seperation axioms which is not preserved under a strengthening of topology.

In this section we provide each of following axiom as an axiom weaker than T_1, but with normality that is stronger than T_4. All of following axioms are based on the observation that in a T_1-space the point closure of a point x is x itself.

Let X be a topological space.

AXIOM T_α: (λ) *Let* $x \in X$, *if* $\bar{x} \neq X$ *then there exists at least a point* a *such that* $a \in X-\bar{x}$, $\bar{a}=a$.

(ν) X *contains at least one element* x *such that* $\bar{x}=x$.

(μ) *If* C, D *are two disjoint closed sets of* X, *and have disjoint neighborhoods, then* $\bar{x}=x$ *for each* $x \in C \cup D$.

AXIOM T_β: (λ') *Let* C *be a closed set of* X, *if* $C \neq X$ *then there exists a non* *void closed set D such that* $C \cap D \neq \phi$ *(void set)*

(ν') *X contain at least one element x such that* $\bar{x} \neq X$.

(μ) *If C, D are two disjoint closed sets of X, and have disjoint neighborhoods, then* $\bar{x} = x$ *for each* $x \in C \cup D$.

AXIOM T_a: (λ'') $\bar{x}=x$ *or* $\bar{x}$ *contains at least three elements, for each* $x \in X$.

(ν'') *X contains at most one element x such that* $\bar{x} \neq x$.

AXIOM T_b: (λ)……………………………………(λ *is identical with* λ *in* T_α).

(ν'') ……………………………………(ν'' *is identical with* ν'' *in* T_a).

(μ)……………………………………(μ *is identical with* μ *in* T_α).

It is clear that axioms T_α, T_β, T_a and T_b are weaker than T_1. Let X be an aggregate of three elements $\{a, b, c\}$, we take ϕ, $\{a\}$, $\{b\}$, $\{a, b\}$, $\{a, b, c\}$ as the familly of closed sets of X, then X is a T_α, T_β, T_a and T_b space but it is not T_1.

PROPOSITION 1. *A normal* T_α *space is a* T_4 *space.*

PROOF. Let X be a normal T_α space. We shall show bellow a normal T_α —space is a T_1-space hence T_4.

If X has one or two elements then it is clearly a T_1 space by (λ, ν). If X contains more than two elements then by (λ, ν), X contains at least two elements a, b such that $\bar{a}=a$, $\bar{b}=b$. Let $B=\{x : \bar{x}=x, \ x \in X\}$. We shall show that $B=X$. Suppose $B \neq X$. Then there is an element x such that $\bar{x} \neq x$. We prove bellow it leads to a contradiction. We can consider following three cases for $\bar{x}$:

(1) $\bar{x}=X$.

(2) $\bar{x} \neq X$, $\bar{x} \supset B$.

(3) $\bar{x} \neq x$, $\bar{x} \neq X$, $\bar{x} \not\supset B$.

(1) Assume $\bar{x}=X$, since there exist two elements $a, b \in B$ such that $\bar{a}=a$, $\bar{b}=b$, by normality there exist two open sets G_a and G_b such that $G_a \cap G_b=\phi$, $G_a \ni a$, $G_b \ni b$. Then one of G_a, G_b does not contain x, hence generally we can assume G_a does not contain x. Let G_a' be complemented set of G_a then G_a' is a closed set and $G_a' \ni x$. But $x \in G_a'$ implies $\bar{x} \subset G_a'$, it contradicts to $\bar{x} = X$. Hence case (1) can not arise.

(2) Assume $\bar{x} \neq X$, $\bar{x} \supset B$ as case one we have a closed set G_a such that $G_a' \supset \bar{x}$ and $a \notin G_a'$ where $a \in B$. It contradicts to $\bar{x} \supset B$. We have also case (2) can

not arise.

(3) Assume $\bar{x} \neq X$, $\bar{x} \not\supset B$, $\bar{x} \neq x$. Then there exists an element a such that $a \notin \bar{x}$, and $a \in B$. Then $\bar{a}$ and $\bar{x}$ are two disjoint closed sets, and by normality $\bar{a}$ and $\bar{x}$ have disjoint neighborhoods, hence by (ν) we have $\bar{x}=x$. It contradicts to $\bar{x} \neq x$. Hence case (3) also can not arise. By above proof we have $B=X$ that is $\bar{x}=x$ for all $x \in X$. Hence we proved a normal T_α - space is T_4-space. Under normallty (λ', μ', ν) implies (λ, ν) hence we have following proposition.

PROPOSITION 2. *A normal* T_β *space is a* T_4 *space.*

In above proof of two propositions, it seems that (μ) has a very important role, but by the proof of a normal T_a-space is a T_4-space, we want to show that conditions (λ), (λ') and (λ'') have fundamentai role in any axiom.

PROPOSITION 3. *A normal* T_a *space is a* T_4 *space.*

PROOF. Let X be a normal T_a space. If X contains only one or two elements then by (λ'', ν'') it is cleary a T_1-space. Let X have more than two elements, suppose X does not satisfy T_1. Then by (λ''), X contains only one element a such that $\bar{a} \neq a$. Let $B=X-a$. If there exist open sets G_α such that $G_\alpha \ni b_\alpha$, $G_\alpha \not\ni a$ for all $b_\alpha \in B$ then B is an open set hence $\bar{a}=a$.

If there exists an element $b \in B$ such that G is open and $G \ni b$ implies $G \ni a$. Then by normality $b_\alpha \in B$ and $b_\alpha \neq b$ implies there is an open set G_α such that $G_\alpha \ni b$ and $G_\alpha \not\ni a$. Hence $B-b=X-\{a, b\}$ is an open set, in other words, $\{a,b\}$ is a closed set. Hence $\bar{a}=a$ or $\bar{a}=\{a, b\}$ but by (λ'') $\bar{a} \neq \{a, b\}$ that is $\bar{a}=a$.

We proved there is no element such that $\bar{a} \neq a$. It follows a normal T_a-space is a T_4-space.

Combinig the proofs of proposition 1 and 3 of this section, we have following proposition.

PROPOSITION 4. *A normal* T_b *space is a* T_4 *space.*

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REFERENCES

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[2] J L. Kelly; *General Topology*, New York, 1955.