COMPLEMENT OF A CONGRUENCE RELATION IN A MODULAR LATTICE

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A congruence relation in a lattice is a binary relation satisfying reflexivity, symmetry, transitivity and substitution.

Let Φ be the lattice of congruence relations of a modular lattice. In this paper, we shall consider a necessary and sufficient conditions in order that a congruence relation has its complemented element in Φ.

Let L be a lattice. The set N of quotients of L is called *quotient ideal* if and only if N satisfies the followings,

(i) For any $a \in L$, $[a, a] \in N$,

(*ii*) For any $[a, b] \in N$, $[x, y] < [a, b]$ implies $[x, y] \in N$,

(iii) If $[a, b] \in N$ and $[a, b]$, $[x, y]$ are projective then $[x, y] \in N$, and

(iv) $[a, b]$, $[b, c] \in N$ implies $[a, c] \in N$.

For any congruence relation θ, a quotient $[a, b]$ is called nullized by θ if $a \equiv b(\theta)$.

Mayeda [2] has proved that given a congruence relation θ on a lattice, let $N(\theta)$ be the set of all quotients nullized by θ, then $N(\theta)$ is a quotient ideal, and conversely given any quotient ideal N, a congruence relation $\theta(N)$ is defined by writing $a \equiv b(\theta(N))$ if and only if $[a \cap b, a \cup b] \in N$. It follows clearly that $N(\theta(N)) = N$ and $N(\theta) < N(\phi)$ if and only if $\theta < \phi$ in Φ.

Let L be a lattice. L is said to *be alternate* for θ if, for each proper quotient $[a, b]$, there exists a finite chain $a = x_0 < x_1 < \cdots\cdots < x_n = b$ such that $x_{i-1} \equiv x_i(\theta)$ and $u \not\equiv v(\theta)$ for any distinct elements $u, v \in [x_i, x_{i+1}]$ alternatively.

The following lemma will be needed.

LEMMA. *Let L be a lattice and θ, ϕ two congruence relations on L. $x \equiv y(\theta \cup \phi)$ if and only if there exists a finite chain $x \cap y = a_0 < a_1 < \cdots\cdots < a_n = x \cup y$ such that $a_i \equiv ai+1(\theta$ or $\phi)$.*

PROOF. The sufficiency is trivial, we shall prove the necessity. Suppose $x \equiv y(\theta \cup \phi)$. Then clearly $x \cap y \equiv x \cup y\ (\theta \cup \phi)$, i.e., we can find a finite sequence $x \cap y = b_0,\ b_1, \cdots\cdots,\ b_n = x \cup y$ such that $b_i \equiv b_{i+1}(\theta$ or $\phi)$. Setting $x_i = [(x \cap y) \cup b_i]$

$\cap [x \cup y]$ $(i=0, 1, \cdots, n)$, clearly $b_i \equiv b_{i+1}(\theta)$, (ϕ) implies $x_i \equiv x_{i+1}(\theta)$, (ϕ), respectively. And we see $x \cap y = x_0 < x_1$. But since $x_1 \equiv x \cup y(\theta \cup \phi)$ and $x_1 \leq x \cup y$, taking x_1 instead of $x \cap y$ in $x \cap y \equiv x \cup y(\theta \cup \phi)$ we can repeat the above process.

Now we prove the main theorem.

THEOREM. *Let L be a modular lattice and θ a congruence relation on L. θ has its complement θ' if and only if L is alternating for θ.*

PROOF. We first prove the sufficiency. Let N' be the set of all quotients $[a, b]$ such that either $a=b$ or $[c, d] \notin N(\theta)$ for any $[c, d] \leq [a, b]$. Then N' is a quotient ideal. In fact, (i) and (ii) conditions are trivial. For (iii), suppose $[a, b] \in N'$ and $[a, b]$, $[x, y]$ are transpose. If $[x, y] \notin N'$, then we can find a proper quotient $[u, v] \in N(\theta)$ such that $[u, v] < [x, y]$. Since $[a, b]$, $[x, y]$ are transpose, we have either $a \cap y = x$ and $a \cup y = b$ or $x \cap b = a$ and $x \cup b = y$. Say $a \cap y = x$ and $a \cup y = b$. By modularity $[u, v]$, $[u \cup a, v \cup a]$ are transpose. It follows $[u \cup a, v \cup a] \in N(\theta)$. But $[u \cup a, v \cup a] \leq [a, b]$ and $[a, b] \in N'$ which is contrary. Hence $[x, y] \in N'$. For (iv), suppose $[a, b]$, $[b, c] \in N'$. If $[a, c] \in N'$ then we can also find a proper quotient $[u, v] \in N(\theta)$ such that $[u, v] \leq [a, c]$. Setting $w = v \cap (u \cap b)$ we have $[w, v] \leq [u, v]$ which follows $[w, v] \in N(\theta)$. It is easily seen that $[w, v]$, $[u \cup b, v \cup b]$ are transpose. Therefore $[u \cup b, v \cup b] \in N(\theta)$. But $[u \cup b, v \cup b] \leq [b, c]$ and $[b, c] \in N'$ which is contrary. Hence $[a, c] \in N'$.

From this quotient ideal N' of L, we can have the congruence relation $\theta(N')$ on L. Now we see $\theta(N')$ is a complement of θ. In fact, for $x \neq y$ in L if $x \equiv y(\theta)$ then $[x \cap y, x \cup y] \in N$, i.e., $[x \cap y, x \cup y] \notin N'$. Therefore $x \not\equiv y(\theta(N'))$. Hence $\theta \cap \theta(N') = 0$. Since L is alternating, for any two distinct elements $x, y \in L$ there exists a finite chain $x \cap y = a_0 < a_1 < \cdots\cdots < a_n = x \cup y$ such that $a_{i-1} \equiv a_i(\theta)$ and $u \not\equiv v(\theta)$ for any distinct elements $u, v \in [a_i, a_{i+1}]$ alternatively. It easily follows that $x \cap y \equiv x \cup y(\theta \cup \theta(N'))$. Hence $\theta \cup \theta(N') = I$.

Now we prove the necessity. Suppose there exists a complement θ' of θ. For each proper quotient $[a, b]$ $a \equiv b(\theta \cup \theta')$. By lemma, there exists a finite chain $a = a_0 < a_1 < \cdots\cdots < a_n = b$ such that $a_i \equiv a_{i+1}(\theta \text{ or } \theta')$. By cancellation of repeating terms we can choose that $a = x_0 < x_1 < \cdots\cdots x_n = b$ so that $x_{i-1} \equiv x_i(\theta)$ and $x_i \equiv x_{i+1}$ (θ') alternatively. And for any distinct elements $u, v \in [x_i, x_{i+1}]$ we can easily see that $u \not\equiv v(\theta)$. Hence L is alternating for θ.

COROLLARY. *Let L be a modular lattice, Φ is a Boolean algebra if and only if L is alternating for any $\theta \in \Phi$.*

COROLLARY. *If L is a modular lattice in which any bounded chain is finite, then Φ is a Boolean algebra.*

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