AN ISOLATED POINT IN A COMPLETE LATTICE WITH ODER TOPOLOGY

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In this paper we obtain a necessary and sufficient condition for an element of a complete lattice to be isolated in the oder topology. G. Birkoff has posed the problem; which should read *"Find a necessary and sufficient condition for an element of a complet lattice to be isolated* (a) *in the order topology,* (b) *in the interval topology."* (page 62 of [1]) The part (b) of the problem has been solved previously by E.S. Northam [2] and by the author [3]. This paper supplies a more or less complete answer to the part (a) of this problem. We here recall some standard terms. L is a complete lattice if it is a partialy ordered, and has supremum and infimum of every subset of L, Let $\{x_\alpha\}$ be any net in a complete lattice L. We define x_α order-converges to a to mean

$$\text{Lim sup }\{x_\alpha\}=\text{ Lim inf }\{x_\alpha\}=a$$

that is, it means

$$a=\bigwedge_\alpha\Big\{\bigvee_{\beta\geqq\alpha} x_\beta\Big\}=\bigvee_\alpha\Big\{\bigwedge_{\beta\geqq\alpha} x_\beta\Big\}$$

And we define a subset M of L to be *closed* in the *order topology,* if and only if, for any net $\{x_\alpha\}$ in M, if x_α order converge to a then $a\epsilon M$. An element a in a complete lattice L is said to have finite property if for every subset S of L such that $a=\vee S$ there exists a finite subset F of S such that $a=\vee F$ and dual.

We are now in a position to prove the following theorem:

THEOREM I. *A necessary and sufficient condition for an element a of a complete lattice to be isolated in the order topology is that a has finite property.*

PROOF. At first we shall show that the condition is necessary. Suppose that for some subset S of the complete lattice L, $a\neq\vee F$ for every finite subset F of S. Let Γ be the set of all finite subset of S. We then know that Γ is a directed set under set inclusion relation. To obtain a net in $L-a$ which order converges to a, we set $b_F=\vee F$ for every $F\epsilon\Gamma$. And we shall show the net $\{b_F\}$ in $L-a$ orderconverges to a. In fact, since S is the set union of all $F\epsilon\Gamma$ we have

$$a=\vee S=\vee_{F\epsilon\Gamma}(\vee F)=\vee_{F\epsilon\Gamma}b_F.$$

Therefore $b_F\uparrow a$ which means that $\{b_F\}$ is a monotone increasing net and

$a=\bigvee_{F\epsilon\Gamma} b_F$. Hence $\{b_F\}$ order-converges to a. Thus it follows that a is not an isolated point of L, which is a contradiction. Hence we have a finite subset F of S such that $a=\bigvee F$. And dual.

We now show the sufficiency. Suppose that $L-a$ is not closed in the order topology, then it means that there exists a net $\{x_\alpha\}$ in $L-a$ such that $\{x_\alpha\}$ order-converges to a, i.e.,

$$a=\bigwedge_\alpha\left\{\bigvee_{\beta\geqq\alpha} x_\beta\right\}=\bigvee_\alpha\left\{\bigwedge_{\beta\geqq\alpha} x_\beta\right\}.$$

Setting $u_\alpha=\bigvee_{\beta\geqq\alpha} x_\beta$ and $v_\alpha=\bigwedge_{\beta\geqq\alpha} x_\beta$ we see that $u_\alpha\downarrow_a$ and $v_\alpha\uparrow_a$ such that $u_\alpha\geqq x_\alpha\geqq v_\alpha$ for all α. Since the element a has the finite property, and $a=\bigvee_\alpha v_\alpha$, there exists a finite subset of the indices set $\{\alpha\}$ of $\{x_\alpha\}$ such that $a=\bigvee_\gamma v_\gamma$. Since $\{\alpha\}$ is a directed set, we can find $\delta\epsilon\{\alpha\}$ such that $\delta\geqq\gamma$ for any $\gamma\epsilon\{\gamma\}$. Since $\{v_\alpha\}$ is monotone increasing, $v_\delta\geqq v_\gamma$ for any $\gamma\epsilon\{\gamma\}$. It follows that $v_\delta=a$. And dualy we have $\sigma\epsilon\{\alpha\}$ such that $v_\sigma=a$. Taking $\tau\epsilon\{\alpha\}$ such that $\tau\geqq\delta,\ \sigma$, we have

$$a=u_\sigma\geqq u_\tau\geqq x_\tau\geqq v_\tau\geqq v_\sigma=a$$

Hence $a=x_\tau$ for some $\tau\epsilon\{\alpha\}$, which is a contradiction.

We have given an obvious corollary:

COROLLARY. *Let L be a complete atomic lattice with unique complement. L is a discrete space in its order topology if and only if the set of all atoms is finite.*

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