ON THE COMPLETENESS OF UNIFORM SPACES

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§ 1. Introduction

Kelley's conjecture* on the completeness of uniform spaces is as follows: a uniform space satisfying the first axiom of countability would be complete if every Cauchy sequence in the space converged to a point of the space.

The main purpose of this note is to prove that his conjecture is false. In § 3 we shall construct a uniform space satisfying the first axiom of countability in which every Cauchy sequence converges to a point but some Cauchy net does not coverge. Such a space evidently is not complete.

I owe tnanks to professor Chi Young Kim who has suggested many improvements for this note.

§ 2 Definitions and theorems.

As a preparation to the following section we shall describe some definitions and theorems which can be found in [1].

Let D be a directed set with the binary relation $\geqq$.

DEFINITION 1. A net $\{x_n | n \in D\}$ is eventually in a set A iff there is an element $m \in D$ such that, if $n \in D$ and $n \geqq m$, then $x_n \in A$.

DEFINITION 2. A net $\{x_n | n \in D\}$ in the uniform space $(X, \mathscr{U})$ is a Cauchy net iff the net $\{(x_m, x_n) | (m, n) \in D \times D\}$ is eventually in each member of the uniformity $\mathscr{U}$. (It is understood that $D \times D$ is given the product ordering.)

DEFINITION 3. A uniform space is complete iff every Cauchy net in the space converges to a point of the space.

THEOREM 1. A family $\mathscr{B}$ of subsets of $X \times X$ is a base for some uniformity for X if and only if

(a) each member of $\mathscr{B}$ contains the diagonal Δ;

(b) if $U \in \mathscr{B}$, then U^{-1} contains a member of $\mathscr{B}$;

* cf. J. L. Kelley "General Topology" (1955) page 193

(c) if $U \in \mathscr{B}$, then $V \circ V \subset U$ for some V in $\mathscr{B}$; and

(d) the intersection of two members of $\mathscr{B}$ contains a member.

THEOREM 2. If $\mathscr{B}$ is a base for the uniformity $\mathscr{U}$ for X, then for each $x \in X$ the family of sets $V[x]$ for V in $\mathscr{B}$ is a base for the neighborhood system of x.

§ 3 Lemmas and main theorem.

Now in this section it will be shown that the Kelley's conjecture is false.

Let X be any uncountable set. For each sequence $S = \{x_i | i \in \omega\}$ in X we make a subset U_s of $X \times X$ such that

$$U_s = (X - \bigcup_i \{x_i\}) \times (X - \bigcup_i \{x_i\}) \cup (\sum_i (x_i, x_i))$$

LEMMA 1. *The subset U_s of $X \times X$ has the following properties.*

(1) *U_s contains the diagonal Δ.*

(2) $U_s = U_s^{-1}$.

(3) $U_s \circ U_s = U_s$.

PROOF. (1) and (2) are clear.

For every member $(x, y) \in U_s \circ U_s$ there exists some point z in X such that $(x, y) = (x, z) \circ (z, y) \in U_s \circ U_s$.

(i) If x or y is a member of the sequence S, then $x = z = y$ and therefore (x, y) belongs to U_s

(ii) If both x and y are not members of the sequence S, then the (x, y) clearly belongs to U_s

It follows from (i) and (ii) that $U_s \circ U_s$ is contained in U_s.

While $U_s \circ U_s \supset U_s \circ \Delta = U_s$, therefore $U_s \circ U_s = U_s$. This establishes (3).

For every sequence S' in X we may construct a subset $U_{s'}$ of $X \times X$ as before.

LEMMA 2. *The family $\mathscr{B} = \{U_s | S$ is a sequence in $X\}$ is a base for some uniformity $\mathscr{U}$ of X.*

PROOF. By Lemma 1 the following conditions (1), (2) and (3) are clearly satisfied in $\mathscr{B}$.

(1) Each member of $\mathscr{B}$ contains the diagonal Δ.

(2) For each U_s in $\mathscr{B}$, $U_s = U_s^{-1}$.

(3) For each U_s in $\mathscr{B}$, $U_s \circ U_s = U_s$.

For arbitrary two members $U_{s'}$ and $U_{s''}$ of $\mathscr{B}$ there are two sequences $S'=\{x_1', x_2', \cdots, x_i', \cdots\}$ and $S''=\{x_1'', x_2'', \cdots, x_i'', \cdots\}$ in X.

Let $S=\{x_1', x_1'', x_2', x_2'', \cdots x_i', x_i'', \cdots\}$. Then since S is a sequence in X there is a member U_s in $\mathscr{B}$. It is clear that $U_{s'} \cap U_{s''} = U_s$. Hence $U_{s'} \cap U_{s''}$ is a member of $\mathscr{B}$. We now have the following result.

(4). The intersection of two members of $\mathscr{B}$ is again a member of $\mathscr{B}$.

(1), (2), (3) and (4) are the sufficient conditions for the family $\mathscr{B}$ to be a base for some uniformity $\mathscr{U}$ of X (by Theorem 1).

Then we have:

LEMMA 3. *$(X, \mathscr{U})$ is a uniform space with the discrete uniform topology.*

PROOF For an arbitrary point x of X there exists a sequence $S'=\{x_i' \mid i \in \omega\}$ in X such that $x=x_i'$ for each $i \in \omega$. Since $U_{s'}$ is a member of the base $\mathscr{B}$, $U_{s'}[x]=\{x\}$ is a neighborohod of x (by Theorem 2). Hence Lemma 3 follows.

LEMMA 4. *A sequence $S=\{x_i \mid i \in \omega\}$ in $(X, \mathscr{U})$ is a Cauchy sequence if and only if there is a $k \in \omega$ such that $x_m=x_n$ for every $m, n \geqq k$.*

PROOF Let $S=\{x_i \mid i \in \omega\}$ be a sequence in $(X, \mathscr{U})$. If there is not $k \in \omega$ satisfying the given condition, then for every $j \in \omega$ there exist $m, n \in \omega$ such that $m, n \geqq j$ and $x_m \neq x_n$. Since x_m and x_n are the members of the sequence S, (x_m, x_n) does not belong to the U_s.

Therefore $S \times S = \{(x_i, x_j) \mid (i, j) \in \omega \times \omega\}$ is not eventually in U_s. Hence the sequence S is not a Cauchy sequence in $(X, \mathscr{U})$. This establishes half of Lemma 4, and the converse is obvious.

LEMMA 5. *Every Cauchy sequence in $(X, \mathscr{U})$ converges to one point of the space.*

PROOF. It is clear by Lemma 4.

Then we have the following main theorem.

MAIN THEOREM. *For the uniform space $(X, \mathscr{U})$ which is constructed as above,*

(a) *the first axiom of countability is satisfied,*

(b) *every Cauchy sequence in $(X, \mathscr{U})$ converges to one point of the space,*

and (c) *there is some Cauchy net in $(X, \mathscr{U})$ which does not converge to a point of the space.*

PROOF. (a) and (b) are obvious by the Lemmas 3 and 5. Let $X_s = X - \{x_i \mid x_i \in S\}$ where $S = \{x_i \mid i \in \omega\}$ is a sequence in X. Then the family $\mathfrak{A} = \{X_s \mid U_s \in \mathscr{B}\}$ is clearly directed by $\subset$, and every member X_s in $\mathfrak{A}$ is a non-empty subset of the set X because the set X is uncountable. For each X_s in $\mathfrak{A}$ we may choose a point y_s in X_s. Then the net $\{(y_{s'}, y_s) \mid (X_{s'}, X_s) \in \mathfrak{A} \times \mathfrak{A}\}$ is eventually in each member of the base $\mathscr{B}$ for the uniformity $\mathscr{U}$, because for each member U_s in $\mathscr{B}$ there is a member X_s in $\mathfrak{A}$ such that $(y_{s''}, y_{s'}) \in U_s$ whenever $X_{s''}, X_{s'}$ follow X_s in the ordering $\subset$. Since the net $\{(y_{s'}, y_s) \mid (X_{s'}, X_s) \in \mathfrak{A} \times \mathfrak{A}\}$ is eventually in each member of the base for the uniformity $\mathscr{U}$, the net $\{y_s \mid X_s \in \mathfrak{A}\}$ is a Cauchy net in $(X, \mathscr{U})$ (by Definition 2).

It is enough to prove that the Cauchy net $\{y_s \mid X_s \in \mathfrak{A}\}$ does not converge to a point of the space. For an arbitrary member X_s in $\mathfrak{A}$ there is a point y_s in X_s which is a member of the Cauchy net.

Let $X_{s'} = X_s - y_s$. Then $X_{s'}$ is a member of $\mathfrak{A}$ and follows X_s. Therefore there is a point $y_{s'}$ in $X_{s'}$ which is a member of the Cauchy net, and $y_{s'} \neq y_s$. This shows that for every point x of X, the Cauchy net is not eventually in $\{x\}$. Since $(X, \mathscr{U})$ is a discrete space by Lemma 3 the Cauchy net can not converge to a point of the space. (c) follows.

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REFERENCES

[1] John L. Kelley: *General topology* (1955)

[2] Sierpinski: *General topology* (1952)

[3] Hausdorff: *Set theory* (1957)

[4] E. Kawano: *Theory of topological spaces* (1957)