NOTE ON INFINITESIMAL TRANSFORMATION IN NORMAL CONTACT SPACE

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Introduction. Recently, S. Sasaki defined the notion of (ϕ, ξ, η, g)-structure of a differentiable manifold[1], and he and Y. Hatakeyama studied its geometric properties[2]. Furthermore, S. Sasaki, Y. Hatakeyama and M. Okumura defined a normal contact space and discussed many interesting theorems in this space[3]. In this space, problems concerning infinitesimal transformations were studied by M. Okumura and S. Tano[4], [5].

In the present note we investigate the relations among several infinitesimal transfomations.

In section 1, we state the fundamental properties of the normal contact space and the definitions of these infinitesimal transformations as the preparation of this note.

We show the results of this note in section 2 and prove them in section 3 successively.

1. Preliminaries. On an $N(=2n+1)$-dimensional real differentiable manifold, if there exist a tensor field ϕ_j^i, contravariant and covariant vector fields ξ^i and η_i satisfying the relations

$$(1.1) \qquad \xi^i \eta_i = 1,$$

$$(1.2) \qquad \text{rank } \|\phi_j^i\| = n-1,$$

$$(1.3) \qquad \phi_j^i \xi^j = 0,$$

$$(1.4) \qquad \phi_j^i \eta_i = 0,$$

$$(1.5) \qquad \phi_j^i \phi_k^j = -\delta_k^i + \xi^i \eta_k,$$

then we call the notion $(\phi_j^i, \xi^i, \eta_j)$ a (ϕ, ξ, η)-structure and the manifold a manifold with a (ϕ, ξ, η)-structure. It is well known that a manifold with a (ϕ, ξ, η)-structure always admits a positive definite Riemannian metric tensor g_{ji} such that

(1.6) $$g_{ji}\xi^j=\eta_i,$$

(1.7) $$g_{ji}\phi^j_k\phi^i_h=g_{kh}-\eta_k\eta_h.$$

We call such a notion (ϕ^i_j, ξ^i, η_j, g_{ji}) satisfying above properties a (ϕ, ξ, η, g)-structure and the manifold a manifold with a (ϕ, ξ, η, g)-stucture. In this note, we always consider such a Riemannian metric tensor, and thus we use a notation η^i in stead of ξ^i.

Next, let M be a differentiable manifold with a contact structure

$$\eta=\eta_i\,dx^i.$$

If we define ϕ_{ji} by

$$2\phi_{ji}=\partial_j\eta_i-\partial_i\eta_j,\quad (\eta_i=\partial/\partial x^i),$$

we can introduce a Riemannian metric g_{ji} such that

(1.8) $$\phi^h_i=g^{hr}\phi_{ir},$$

(1.9) $$\xi^i=g^{ir}\eta_r,$$

where η_i, g_{ji}, ϕ^i_j define a (ϕ, ξ, η, g)-structure.

S. Sasaki and Y. Hatakeyama introduce four important tensor fields $N_{ji}{}^h$, N_{ji}, N^j_i and N_j, and if $N_{ji}{}^h$ vanishes, the other three tensors vanish. We call the contact space with vanishing $N_{ji}{}^h$ a normal contact space.

In a normal contact space, the following identities are always valid[3],

(1.10) $$\nabla_j\eta_i=\phi_{ji},$$

(1.11) $$\nabla_k\phi_{ji}=\nabla_k\nabla_j\eta_i=\eta_j g_{ik}-\eta_i g_{jk},$$

(1.12) $$\eta_r R_{lkj}{}^r=\eta_l g_{jk}-\eta_k g_{jl},$$

(1.13) $$\nabla_l\nabla_k\phi_{ji}=\phi_{lj}g_{ik}-\phi_{li}g_{jk}.$$

(1.14) $$-\phi_{jr} R_{lki}{}^{r} - \phi_{ri} R_{lkj}{}^{r} = \phi_{lj} g_{ik} - \phi_{kj} g_{il} - \phi_{li} g_{jk} + \phi_{ki} g_{jl},$$

where $R_{lkj}{}^{i}$ is a Riemannian curvature tensor, and ∇_j is the differentiation with respect to Riemannian connection.

In a Riemannian space, if the vector field v^i satisfies

(1.15) $$\pounds_v g_{ji} \equiv \nabla_j v_i + \nabla_i v_j = 0,$$

(1.16) $$\pounds_v g_{ji} \equiv \nabla_j v_i + \nabla_i v_j = 2\sigma g_{ji},$$

(1.17) $$\pounds_v \left\{ {h \atop ji} \right\} \equiv \nabla_j \nabla_i v^h + R_{rji}{}^{h} v^r = 0,$$

and

(1.18) $$\pounds_v \left\{ {h \atop ji} \right\} \equiv \nabla_j \nabla_i v^h + R_{rji}{}^{h} v^r = \rho_j \delta_i^h + \rho_i \delta_j^h,$$

then the vector v^i is called respectively an infinitesimal isometry (or Killing vector), an infinitesimal conformal transformation (or conformal Killing vector), an infinitesimal affine collineation and an infinitesimal projective transformation, where σ is a certain scalar function called by an associated scalar of the transformation and ρ_j is a certain vector field called by an associated vector of the transformation. If v^i admits a conformal Killing, then it holds

(1.19) $$\pounds_v \left\{ {h \atop ji} \right\} \equiv \nabla_j \nabla_i v^h + R_{rji}{}^{h} v^r = \sigma_j \delta_i^h + \sigma_i \delta_j^h - \sigma^h g_{ji}, \quad (\sigma_i = \partial_i \sigma).$$

Next, let us recall the identities of Lie derivatives. For any vector field v^i and tensor field $T_{ji}{}^{h}$, we have the following identities:

(1.20) $$\pounds_v T_{ji}{}^{h} = v^a \nabla_a T_{ji}{}^{h} + T_{ai}{}^{h} \nabla_j v^a + T_{ja}{}^{h} \nabla_i v^a - T_{ji}{}^{a} \nabla_a v^h,$$

(1.21) $$\pounds_v \nabla_m T_{ji}{}^{h} - \nabla_m \pounds_v T_{ji}{}^{h} = -T_{ai}{}^{h} \pounds_v \left\{ {a \atop jm} \right\} - T_{ja}{}^{h} \pounds_v \left\{ {a \atop im} \right\} + T_{ji}{}^{a} \pounds_v \left\{ {h \atop am} \right\},$$

(1.22) $$\nabla_k \pounds_v \left\{ {h \atop ji} \right\} - \pounds_v \nabla_k \left\{ {h \atop ji} \right\} = \pounds_v R_{kji}{}^{h}.$$

In a normal contact space, if the vector field v^i satisfies

(1.23) $\pounds_v \eta_i = \tau \eta_i$,

and

(1.24) $\pounds_v \phi^i_j = 0$,

then we call the vector v^i an infinitesimal contact transformation (τ is an associated scalar of the transformation) and *an infinitesimally ϕ-invariant transformation* repectictively, and especially, when τ vanishes v^i is called by an infinitesimal strict contact transformation.

Transvecting (1.23) with η^i, we have

(1.25) $\tau = \eta^i \pounds_v \eta_i = \eta^c \eta^b \nabla_c v_b$.

2. Results. In this section, we summarize the results of the present note as followings:

(I). *In a contact space, if an infinitesimal contact transformation admits an infinitesimal isometry, then it is strict and the length of the vector field η^i is invariant under the infinitesimal isometry.*

Next, we consider the normal contact space, and then, do not refer the space in the followings.

(II). *If an infinitesimal isometry v^i is infinitesimally ϕ-invariant, then it admits an infinitesimal strict contact transformation and the length of the vector v^i is constant along the curve tangented to direction η^i.*

(III). *If an infinitesimal affine collineation is infinitesimally ϕ-invariant, then it admits an infinitesimal strict contact transformation and an infinitesimal isometry.*

(IV). *If an infinitesimal projective transformation is infinitesimally ϕ-invariant, then it admits a conformal Killing.*

(V). *If a conformal Killing is infinitesimally ϕ-invariant, then it admits an infinitesimal isometry.*

(VI). *If an infinitesimal contact transformation v^i is infinitesimally ϕ-invariant, then its associated scalar τ is represented by*

(2.1) $\tau = \frac{2}{N+1} \nabla_a v^a$.

(VII). *If an infinitesimal contact transformation admits an infinitesimal affine collineation, then its associated scalar τ is represented by* (2.1) *and constant along the curve tangented to direction η^i. If an infinitesimal isometry v^i admits an infinitesimal contact tranformation, or if an infinitesimal affine collineation admits an infinitesimal strict contact transformation, then it holds*

$$(2.2) \qquad \nabla_a v^a = 0.$$

(VIII). *If an infinitesimal contact tranformation v^i admits an infinitesimal projective transformation, then its associated scalar τ is represented by* (2.1) *and a direction $\tau_k - 2\rho_k$ is orthogonal to the direction η^i. Furthermore if the transformat on is strict, then it admits an infinitesimal affine collineation.*

(IX). *If an infinitesimal contact transformation admits a conformal Killing, then it admits an infinitesimal isometry and strict contact transformation.*

3. **Proofs.** In this section, we prove above results successively.

(I). Let v^i be an infinitesimal contact transformation, then we have

$$\pounds_v \eta_j = \tau \eta_j.$$

Since v^i admits an infinitesimal isometry, we have

$$\pounds_v \eta^i = \tau \eta^i.$$

Taking Lie derivative of (1.5), we obtain by means of above

$$\phi^i_j \pounds_v \phi^j_k + \phi^j_k \pounds_v \phi^i_j = 2\tau \eta^i \eta_k.$$

Transvecting this with $\eta^k \eta_i$ we have

$$(3.1) \qquad \tau = 0,$$

and thus, it is strict.

Form (3.1) and (1.25), we have

$$(3.2) \qquad \eta^i \pounds_v \eta_i = 0.$$

Since v^i admits an infinitesimal isometry, we obtain

$$(3.3) \qquad \pounds_v \eta^2 = 0,$$

(II). Taking Lie derivative of (1.5) and v^i being infinitesimally ϕ- invariant, we have

$$(3.4)\qquad \eta^i \pounds_v \eta_k + \eta_k \pounds_v \eta^i = 0.$$

Transvecting this with η_i, we have

$$\pounds_v \eta_k + \eta_k \eta_a \pounds_v \eta^a = 0.$$

Since v^i admits an infinitesimal isometry, it is reduced into

$$(3.5)\qquad \pounds_v \eta^k + \eta^k \eta_a \pounds_v \eta^a = 0.$$

Transvecting this with η_k, we have

$$\eta_a \pounds_v \eta^a = 0.$$

and thus, we obtain that v^i admits an infinitesimal strict contact transformation.

On the other hand, (3.4) is reduced into

$$\eta_k (v^a \nabla_a \eta^i - \eta^a \nabla_a v^i) + \eta^i (v^a \nabla_a \eta_k + \eta_a \nabla_k v^a) = 0.$$

Transvecting this with η^k and taking account of (1.8) and (1.10), we have

$$v^a \phi_{ai} - \eta^a \nabla_a v_i = 0,$$

since v^i is an infinitesimal isometry. Transvecting this with v^i, by virtue of skew-symmetric property of ϕ_{ai}, we have

$$v^j \eta^i \nabla_i v_j = 0,$$

or

$$\eta^a \nabla_a v^2 = 0,$$

and thus, (II) has been proved.

(III). For any vector v^i, it holds

$$(3.6)\qquad \pounds_v \nabla_k \phi^i_j - \nabla_k \pounds_v \phi^i_j = \phi^a_j \pounds_v \left\{ {i \atop ka} \right\} - \phi^i_a \pounds_v \left\{ {a \atop kj} \right\}.$$

From the assumption, we have

$$\pounds_v \nabla_k \phi_j^i = 0.$$

Substituting (1.11) into this, we have

$$\pounds_v (\eta_j \delta_k^i - \eta^i g_{kj}) = 0,$$

and it is reduced into

$$(v^a \nabla_a \eta_j + \eta_a \nabla_j v^a)\delta_k^i - (v^a \nabla_a \eta^i - \eta^a \nabla_a v^i) g_{kj} - \eta^i (\nabla_k v_j + \nabla_j v_k) = 0. \tag{3.7}$$

Contracting with respect to i and k, we have

$$v^a \nabla_a \eta_j + \eta_a \nabla_j v^a = 0, \tag{3.8}$$

or

$$\pounds_v \eta_j = 0,$$

and hence, we have proved former of (III).

Next, in consequence of (3.8), (3.7) is reduced into

$$\eta^i (\nabla_k v_j + \nabla_j v_k) = -(v^a \nabla_a \eta^i - \eta^a \nabla_a v^i) g_{kj}.$$

Transvecting this with η_i, we have

$$\nabla_k v_j + \nabla_j v_k = (\eta^c \eta^b \nabla_c v_b) g_{kj}. \tag{3.9}$$

In consequence of (1.25) and the former of this theorem, we have

$$\nabla_k v_j + \nabla_j v_k = 0,$$

and thus, (III) has been proved.

(IV). From (3.6) and the assumption we have by means of (1.11)

$$(v^a \nabla_a \eta_j + \eta_a \nabla_j v^a)\delta_k^i - (v^a \nabla_a \eta^i - \eta^a \nabla_a v^i) g_{kj} - \eta^i (\nabla_k v_j + \nabla_j v_k) = \phi_j^a (\rho_k \delta_a^i + \rho_a \delta_k^i) - \phi_a^i (\rho_k \delta_j^a + \rho_j \delta_k^a). \tag{3.10}$$

Contracting with respect to i and k, we have, in consequence of $\phi_a^a = 0$,

$$v^a \nabla_a \eta_j + \eta_a \nabla_j v^a = \frac{N}{N-1} \phi_j^a \rho_a.$$

Substituting this into (3.10), we have

$$\frac{N}{N-1}\phi_j^a\rho_a\delta_k^i-(v^a\nabla_a\eta^i-\eta^a\nabla_a v^i)g_{kj}-\eta^i(\nabla_k v_j+\nabla_j v_k)$$
$$=\phi_j^a(\rho_k\delta_a^i+\rho_a\delta_k^i)-\phi_a^i(\rho_k\delta_j^a+\rho_j\delta_k^a).$$

Transvecting this with η_i, we have

$$(3.11)\qquad \nabla_k v_j+\nabla_j v_k=\frac{1}{N-1}\phi_j^a\rho_a\eta_k+g_{kj}\eta^b\eta^a\nabla_b v_a.$$

Interchanging j and k, and subtracting these, we have

$$(\phi_k^a\eta_j-\phi_j^a\eta_k)\rho_a=0.$$

Transvecting this with η^j, we obtain

$$\phi_j^a\rho_a=0.$$

Form this, (3.11) is reduced into

$$\nabla_k v_j+\nabla_j v_k=(\eta^c\eta^b\nabla_c v_b)g_{kj},$$

and thus, we have proved (IV).

(V). From (3.6), we have in consequence of the assumption and (1.19)

$$v^a\nabla_a\nabla_k\phi_j^i+\nabla_a\phi_j^i\nabla_k v^a+\nabla_k\phi_a^i\nabla_j v^a-\nabla_k\phi_j^a\nabla_a v^i$$
$$=-\phi_k^i\sigma_j+g_{kj}\sigma^a\phi_a^i+\delta_k^i\phi_j^a\sigma_a-\sigma^i\phi_{jk},$$

and from (1.11), it is reduced into

$$(3.12)\qquad g_{ik}v^a\nabla_a\eta_j-g_{kj}v^a\nabla_a\eta_i-\eta_i\nabla_k v_j+\eta_a g_{ik}\nabla_j v^a$$
$$-\eta_i\nabla_j v_k+g_{kj}\eta_a\nabla^a v_i$$
$$=-\phi_{ki}\sigma_j+g_{kj}\sigma^a\phi_{ai}+g_{ki}\sigma_a\phi_j^a-\sigma_i\phi_{jk}.$$

Transvecting this with $g^{jk}\eta^i$, we have

$$(3.13)\qquad 2\nabla^a v_a=(N+1)\eta^c\eta^b\nabla_c v_b.$$

On the other hand, since v^i admits a conformal Killing, it holds

$$(3.14)\qquad \nabla_c v_b+\nabla_b v_c=2\sigma g_{cb}.$$

Transvecting this with g^{cb} and $\eta^c \eta^b$, then we obtain respectively

$$\nabla^a v_a = N\sigma,$$

and

$$\eta^c \eta^b \nabla_c v_b = \sigma.$$

Substituting these into (3.13), we have

$$(N-1)\sigma = 0,$$

or

$$\sigma = 0,$$

and thus, we have proved (V).

(VI). For any vector v^i, it holds

$$\text{(3.15)} \qquad \pounds_v \nabla_k \eta_j - \nabla_k \pounds_v \eta_j = -\eta_a \pounds_v \left\{ {a \atop kj} \right\}.$$

By assumption, from (1.10) and (1.17) this is reduced into

$$(\nabla_j v_a + \nabla_a v_j)\phi^a_k - \eta_j \nabla_k \tau - \tau \nabla_k \eta_j = -\eta_a \nabla_k \nabla_j v^a - v^s \eta_a R_{skj}{}^a,$$

since it holds $\nabla_k \eta_j = \phi_{kj} = g_{aj} \phi^a_k$.

Transvecting this with ϕ^{kj}, we obtain from (1.12)

$$(\nabla_j v_k + \nabla_k v_j)(g^{kj} - \eta^k \eta^j) - (N-1)\tau = \phi^{kj} \eta_a \nabla_k \nabla_j v^a.$$

Using of the Ricci formula, we have

$$(N-1)\tau = 2(\nabla^a v_a - \eta^c \eta^b \nabla_c v_b).$$

From (1.25), we have

$$\text{(3.16)} \qquad \tau = \frac{2}{N+1} \nabla_a v^a.$$

(VII). From (3.15), we have by assumption

$$v^a \nabla_a \nabla_k \eta_j + \nabla_a \eta_j \nabla_k v^a + \nabla_k \eta_a \nabla_j v^a - \eta_j \nabla_k \tau - \tau \nabla_k \eta_j = 0,$$

and from (1.10) and (1.11), we have

$$\eta_k v_j - \eta_j v_k + \phi_{aj} \nabla_k v^a + \phi_{ka} \nabla_j v^a - \eta_j \nabla_k \tau - \tau \phi_{kj} = 0. \tag{3.17}$$

Transvecting this with ϕ^{kj}, we have

$$(N-1)\tau = 2(\nabla_a v^a - \eta^c \eta^b \nabla_c v_b).$$

From (1.25), we obtain

$$\tau = \frac{2}{N+1} \nabla_a v^a.$$

And transvecting (3.17) with g^{kj}, we have

$$\eta^a \nabla_a \tau = 0.$$

and thus, the associated scalar τ is constant along the curve tangented to direction η^i.

Further, if v^i admits an infinitesimal strict contact transformation, then we have

$$\nabla_a v^a = 0. \tag{3.18}$$

Next, if v^i admits an infinitesimal isometry, v^i admits an infinitesimal affine collineation and v^i is strict in consequence of (I). Then we obtain (3.18).

(VIII). From (3.15) and (1.11), we have by the assumption

$$\eta_k v_j - \eta_j v_k + \phi_{aj} \nabla_k v^a + \phi_{ka} \nabla_j v^a - \eta_j \nabla_k \tau - \tau \phi_{kj} = -\rho_k \eta_j - \rho_j \eta_k. \tag{3.19}$$

Transvecting this with ϕ^{kj}, we have from (1.25)

$$(N-1)\tau = 2(\nabla_a v^a - \tau),$$

or

$$\tau = \frac{2}{N+1} \nabla_a v^a.$$

And transvecting (3.19) with $\eta^k \eta^j$, we obtain

$$\eta^a(\tau_a - 2\rho_a) = 0, \tag{3.20}$$

and thus, a direction $\tau_k - 2\rho_k$ is orthogonal to the direction η^i.

Next, if the transformation is strict, then it holds

(3.21) $$\rho_a \eta^a = 0,$$

and (3.19) is reduced into

$$\eta_k v_j - \eta_j v_k + \phi_{aj} \nabla_k v^a + \phi_{ka} \nabla_j v^a = -\rho_k \eta_j - \rho_j \eta_k.$$

Interchainging for k and j and summing up these, we have

$$\rho_k \eta_j + \rho_j \eta_k = 0.$$

Transvecting this with η^k, we have by virtue of (3.21)

$$\rho_j = 0,$$

and hence, the transformation admits an infinitesimal affine collineation.

(Ⅸ). From (3.15) and (1.11), we have by the assumption

(3.22) $$\eta_k v_j - \eta_j v_k + \phi_{aj} \nabla_k v^a + \phi_{ka} \nabla_j v^a - \eta_j \nabla_k \tau - \tau \phi_{kj}$$
$$= -\sigma_k \eta_j - \sigma_j \eta_k + \eta_a \sigma^a g_{kj}.$$

Transvecting this with ϕ^{kj}, we have

$$2(\nabla_a v^a - \eta^c \eta^b \nabla_c v_b) = (N-1)\tau.$$

From (1.25), it is reduced into

(3.23) $$(N+1)\tau = 2 \nabla_a v^a.$$

On the other hand, since v^i admits a conformal Killing, it holds

$$\nabla_k v_j + \nabla_j v_k = 2\sigma g_{jk}.$$

Transvecting this by $\eta^k \eta^j$ and g^{kj} we have from (1.25) respectively

(3.24) $$\tau = \sigma,$$

and

(3.25) $$\nabla_a v^a = N\sigma.$$

From (3.23), (3.24) and (3.25), we have

$$(N+1)\tau = 2N\sigma = 2N\tau,$$

or

$$\tau = 0 \quad \text{and} \quad \sigma = 0,$$

and thus, we have proved (Ⅸ).

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