NOTES ON THE LATTICE OF CONGRUENCE RELATIONS ON A LATTICE

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Introduction. Let L be a lattice. A *congruence relation* θ in L is a binary relation $a \equiv b\ (\theta)$ between two elements a and b in L, defined by the four properties:

(1) Reflexivity. For any a, $a \equiv a\ (\theta)$.

(2) Symmetry. When $a \equiv b\ (\theta)$, then $b \equiv a\ (\theta)$.

(3) Transitivity. When $a \equiv b\ (\theta)$ and $b \equiv c\ (\theta)$, then $a \equiv c\ (\theta)$.

(4) Substitution. When $a \equiv b\ (\theta)$ and $x \equiv y\ (\theta)$, then $a \cap x \equiv b \cap y\ (\theta)$ and $a \cup x \equiv b \cup y\ (\theta)$.

Let Φ be the set of all congruence relations defined on L. G. Birkhoff [1] has proved the following theorem: Let C be any subset of Φ. One defines new relations ξ and η by (i) $a \equiv b\ (\xi)$ means $a \equiv b\ (\theta)$ for all $\theta \in C$, (ii) $a \equiv b\ (\eta)$ means that for some finite sequence: $a = x_0,\ x_1,\ \cdots\cdots,\ x_m = b,\ x_{i-1} \equiv x_i\ (\theta_i)$ for some $\theta_i \in C$. Then ξ, η are congruence relations; moreover ξ is the g. l. b. and η the l. u. b. of the C. And N. Funayama and T. Nakayama have proved that Φ satisfies the infinite distributive law: $\theta \cap (\cup_r \theta_r) = \cup_r (\theta \cap \theta_r)$ for all $\theta,\ \theta_r \in \Phi$.

In this paper we shall find a necessary and sufficient conditions that Φ should form a Boolean algebra, when all bounded chains in L are finite.

Cn the other hand, we can see that on even a chain C which is dense-in-itself, the set Φ of all congruence relations is not complemented. In fact, let θ be the congruence relation on C such that $a \equiv b\ (\theta)$ for all $a,\ b \in [c,\ +\infty]$, $x \equiv y\ (\theta)$ for all x, $y \in [-\infty,\ c)$ and $a \not\equiv x\ (\theta)$ for any $a \in [c,\ +\infty]$ and any $x \in [-\infty,\ c)$, where $C = [-\infty, c) \vee [c, +\infty]$. Now suppose that there exists a complement θ' of θ: $\theta \cup \theta' = I$ and $\theta \cap \theta' = O$, where $a \equiv b\ (I)$ holds for all a, b in C, I will be called the *unit* conguence relation, and $a \equiv b (O)$ holds only when $a = b$, O will be called the *zero* congruence relation. Thus we have $c \equiv x\ (\theta \cup \theta')$ for any $x < c$. therefore there exists a sequence $\{x_i\}$ such that $c = x_0,\ x_1,\ \cdots\cdots,\ x_n = x,\ x_{i-1} \equiv x_i\ (\theta \text{ or } \theta')$. Since $c \not\equiv x (\theta)$ for $x < c$ we can find the first $x_j \in \{x_i\}$ such that $x_{j-1} \geqq c > x_j$ and $x_{j-1} \not\equiv x_j\ (\theta)$, and hence $x_{j-1} \equiv x_j\ (\theta')$ and $x_j \equiv c\ (\theta')$ because any congruence class is convex. Since C is dense-in-itself, there exists $y \in C$: $x_j < y < c$. And $x_j \equiv y\ (\theta')$. But x_j, $y \in [-\infty,\ c)$ and $x_j \equiv y\ (\theta)$. Hence $x_j \equiv y\ (\theta \cap \theta')$, which follows $\theta \cap \theta \neq 0$. It means $\theta' \neq$ complement of θ.

According to the above reason, we consider only the case that all bounded chains are finite in a lattice. The main contents of the paper are as follows. In the first section we shall prove that Φ of a modular lattice in which any bounded chains are finite forms a Boolean algebra. J. von Neumann [2] has proved the above statement for a modular lattice of finite length. In the second section for non-modular lattice we shall find a congruence relation which has its complement congruence relation, and find a necessary and sufficient condition that Φ forms a Boolean algebra. In the third section we define an l-congruence relations on an l-group, and we shall prove that if an l-group is satisfing the chain condition, then all l-congruence relations forms a sub-Boolean algebra of the Boolean algebra Φ.

2. Congruence relations on a modular lattice.

In the following we shall use the ordinary terminology of [1]. A closed interval (or quotient) $[x, y]$ is called *prime* if and only if y covers x. Intervals which can be written as $[x \cap y, x]$ and $[y, x \cup y]$ are called *transposes*, while two quotients $[x, y]$ and $[x', y']$ are called *projective* if and only if there exists a finite sequence $[x, y]$, $[x_1, y_1]$, ..., $[x', y']$ in which any two sucessive quotients are transposes, in symbols $[x, y] \sim [x', y']$. And it is well known that the relation of projectivity between prime intervals is an equivalence relation. Hence we can consider the set P of classes $P\alpha$ of projective prime quotients. And $P\alpha$ shall be called *projective* class. In this section we shall prove that Φ of a modular lattice in which all bounded chains are finite forms a Boolean algebra, which is some extension of J. von Neumann's result [2]. First of all, we prove the following two lemmas

LEMMA 1. (i) *Let L be a modular lattice in which all bounded chains are finite, $P\alpha$ a class of projective prime quotients (i.e. $P\alpha \in P$) and θ a congruence relation on L. Then for some $[a, b] \in P\alpha$ if $a \equiv b(\theta)$, then $c \equiv d(\theta)$ for all $[c, d] \in P\alpha$.*

(ii) *Suppose $[a \cap b, a \cup b]$ is a modular sublattice of L. And let $[a \cap b, b]$, $[a, a \cup b]$ be transposes quotients. Then for a maximal chain: $a \cap b < x_1 < x_2 < \cdots x_{n-1} < b$ connecting $a \cap b$ to b, there exists a maximal chain: $a < x'_1 < x'_2 < \cdots < x'_{n-1} < a \cup b$ connecting a to $a \cup b$ such that $[x_i, x_{i+1}] \sim [x'_i, x'_{i+1}]$.*

PROOF. (i) is obvious.

For (ii), we are well known [1] that the correspondence: $x \to a \cup x$ is isomorphism between $[a \cap b, b]$ and $[a, a \cup b]$. Since the chain $a \cap b < x_1 < \cdots < x_{n-1} < b$ is maximal which means x_{i-1} covers x_i, the chain $a < x_1 \cup a < \cdots < x_{n-1} \cup a < b \cup a$ is also maximal. And $[x_i, x_{i+1}] \sim [x_i \cup a, x_{i+1} \cup a]$. In fact,

$$x_{i+1} \cap (x_i \cup a) = x_i \cup (x_{i+1} \cap a) = x_i \cup \{(x_{i+1} \cap b) \cap a\} = x_i \cup (b \cap a) = x_i.$$

LEMMA 2. *Under the same hypotheses of lemma I, for a chain Γ connecting a to b, let S be the set of projective classes having at least one prime quotient in Γ. Then any prime quotient in any chain connecting a to b belongs to some P_α in S.*

PROOF. If chain Γ: $a = x_0 < x_1 < \cdots < x_m = b$ has length m, then by Jordan-Dedekind chain condition every maximal chain cennecting a to b has length m. So let Γ': $a = y_0 < y_1 < \cdots < y_m = b$ be any other maximal chain connecting a to b. Using induction on m, we are going to prove the lemma. If $m=1$, then the lemma is obvious. We assume the lemma holds for all $m \leqq n-1$. For the case $m=n$, if $x_1 = y_1$ then lemma also holds by the hypothese of induction. Suppose $x_1 \neq y_1$. Since x_1 and y_1 cover a, $u = x_1 \cup y_1$ covers x_1 and y_1 by the covering condition. Since each $[x_1, x_2]$, ..., $[x_{n-1}, x_n] \in$ some P_α in S, the hypothese leads also $[x_1, u] \in$ some P_α in S, and so is any prime quotient in $[u, b]$. Because of $[a, x_1] \sim [y_1, u]$, $[y_1, u] \in$ some P_α in S. Hence each $[y_1, y_2]$, ..., $[y_{n-1}, y_n] \in$ some P_α in S. Since $[a, y_1] \sim [x_1, u]$, it follows each $[a, y_1]$, $[y_1, y_2]$, ..., $[y_{n-1}, y_n] \in$ some P_α in S.

Now we prove the following main lemma

LEMMA 3. *Let L be a modular lattice in which all bounded chain are finite, and θ congruence relation on L. Then there exists a complement congruence relation θ' of θ: $\theta \cap \theta' = O$ and $\theta \cup \theta' = I$.*

PROOF. Let S_1 be the set of projective classes having at least one prime quotient which is in the quotient $[a \cap b, a \cup b]$ for some $a \equiv b(\theta)$, and P the set of all projective classes on L. Set $S_2 = P - S_1$. We define a new congruence relation θ' as following: $x \equiv y\ (\theta')$ if and only if either $x = y$ or any prime quoutient in the quotient $[x \cap y,\ x \cup y] \in$ some P_α in S_2. Then we can see that the relation θ' is a congruence relation, moreover $\theta \cap \theta' = O$ and $\theta \cup \theta' = I$. For, from the definition we have directly that θ' is reflexive and symmetric. Next for the transitivity, we suppose $x \equiv y\ (\theta')$ and $y \equiv z\ (\theta')$, i.e. for all prime quotients J in $[x \cap y,\ x \cup y]$ or in $[y \cap z,\ y \cup z]$, $J \in$ some P_α in S_2. Since

$$[x \cap y,\ x \cup y] \supset [x \cap y,\ (x \cap y) \cup (y \cap z)] \sim [x \cap y \cap z,\ y \cap z],$$

by (ii) of lemma 1, we have that all prime quotients $J \in [x \cap y \cap z,\ y \cap z]$ implies $J \in$ some P_α in S_2. Dually, any prime quotient $J \in [y \cup z,\ x \cup y \cup z]$ implies $J \in$ some P_α in S_2. Since

$$\begin{aligned} &[x \cap y \cap z,\ y \cap z] \vee [y \cap z,\ y \cup z] \vee [y \cup z,\ x \cup y \cup z] \\ &\subset [x \cap y \cap z,\ x \cup y \cup z], \end{aligned}$$

we see that there exists a chain γ connecting $x \cap y \cap z$ to $x \cup y \cup z$ such that any prime quotient of $\gamma \in$ some P_α in S_2. Hence by lemma 2 it follows that any prime quotient J in $[x \cap y \cap z,\ x \cup y \cup z]$ is contained in some P_α in S_2. And so is any prime quotient in $[x \cap z,\ x \cup z]$, which follows $x \equiv z\ (\theta')$. Hence the the relation θ' is transitive. Now we will see that the substitute property of θ' holds. It is sufficient to show that

$$x \equiv y\ (\theta') \text{ implies } x \cup z \equiv y \cup z\ (\theta') \text{ and } x \cap z \equiv y \cap z\ (\theta')$$

for any $z \in L$. Suppose $x \equiv y(\theta')$ i.e., for any prime quotient J in $[x \cap y,\ x \cup y] \in$ some P_α in S_2. Since

$$\begin{aligned} &[x \cap y \cap z,\ (x \cap z) \cup (y \cap z)] \subset [x \cap y \cap z,\ (x \cup y) \cap z] \\ &\sim [x \cap y,\ (x \cap y) \cup \{(x \cup y) \cap z\}] \subset [x \cap y,\ x \cup y], \end{aligned}$$

it follows that any prime quotient in

$$[x \cap y \cap z,\ (x \cap z) \cup (y \cap z)] = [(x \cap z) \cap (y \cap z),\ (x \cap z) \cup (y \cap z)]$$

is contained in some P_α in S_2. Hence $x \cap z \equiv y \cap z\ (\theta')$. And dually we see $x \cup z \equiv y \cup z\ (\theta')$. This concludes θ' is a congruence relation on L. Finally we prove that $\theta \cap \theta' = 0$ and $\theta \cup \theta' = I$. In fact, if $a \equiv b(\theta)$ $(a \neq b)$, i.e., any prime quotient in $[a \cap b,\ a \cup b]$ is contained in some P_α in S_1. Since $S_2 = P - S_1$, we have $a \not\equiv b\ (\theta')$. Hence it leads to $\theta \cap \theta' = O$. Next for any $a,\ b \in L$, $[a \cap b,\ a \cup b]$ is a bounded chain. Therefore there exists a maximal chain: $a \cap b < x_1 < x_2 < \cdots < a \cup b$ of finite length. While $x_{i-1} \equiv x_i\ (\theta \text{ or } \theta')$ we have $a \cap b \equiv a \cup b\ (\theta \cup \theta')$. This leads to $a \equiv b\ (\theta \cup \theta')$, which shows $\theta \cup \theta' = I$.

Hence we have the following theorem

THEOREM 1. *Let L be a modular lattice in which all bounded chains are finite. Let $\Phi=\{\theta \mid \theta$ is congruence relation on $L\}$.* Then *Φ forms a Boolean algebra. Moreover $\Phi\cong 2^{P}$, where P is the set of all projective classes of prime qutients on L.*

PROOF. We are well known Φ is distributive, and complemented by the lemma 3. Thus Φ is Boolean algebra. In the lemma 1 and the prove of lemma 3 it is established that for some congrucnce relation θ there exists one and only one subset S of P, and converse. It is evident that $\theta_1 \leqq \theta_2$ if and only if $S(\theta_1) \leqq S(\theta_2)$ in P, where $S(\theta_i)$ is the subset of P corresponding to the congruence relation θ_i.

3. Congruence relations on non-modular lattice.

In this section we consider the set Φ of all congruence relations on non-modular lattice L.

Y. Mayeta [3] has defined the following quotient ideal:

DEFINITION. Let L be a lattice and N the set of qoutients of L is called *quotient ideal* if and only if N satisfies the followings.

(i) For any $a \in L$ $[a, a] \in N$.

(ii) For any $[a, b] \in N$, $[x, y] \subset [a, b]$ implies $[x, y] \in N$.

(iii) For any $[a, b] \in N$, $[a, b]\sim[x, y]$ implies $[x, y] \in N$.

(iv) $[a, b]$, $[b, c] \in N$ implies $[a, c] \in N$.

And for a congruence relation θ a quotient $[a, b]$ is called *nullized* by θ if $a \equiv b$ (θ). He also has proved that if N is a quotient ideal in L, then there exists a congruence relation θ such that N is equal to the set of all quotients which are nullized by the θ. It is well known that if L is non-modular lattice, then L contains a sublattice isomorphice to the five-element lattice of Fig. 1, and a sublattice $M=[z, u]$ in L is called *non-modular subset*. A quotient $[x, y]$ shall be said *non-modular quotient* if there exists a finite sequence $\{z_i\}$: $x=z_0<z_1<\cdots<z_n=y$ such that either

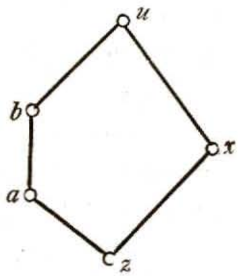

Fig. 1.

$$[z_{i-1}, z_i] \sim [\alpha_i, \beta_i] \qquad \text{or} \quad [z_{i-1}, z_i] \subset [\alpha_i, \beta_i]$$

for α_i, $\beta_i \in$ some non-modular subset of L. A quotient $[a, b]$ shall be said to be *chain connected* to $[c, d]$ when there exists a finite series of quotients $[x_i, y_i]$ such that

$$[a, b] \sim (\text{or } \subset) \ [x_1, y_1] \sim (\text{or } \subset) \ [x_2, y_2] \sim (\text{or } \subset) \cdots \sim (\text{or } \subset) \ [c, d].$$

Now let N be the set of all quotients $[a, b]$ which is chain connected to a quotient $[c, d]$ having a finite series of quotients $[c, u_1]$, $[u_1, u_2], \cdots, [u_{m-1}, d]$ such that after finite chain connecting steps each $[u_{i-1}, u_i]$ is finally chain connected to some non-modular quotient.

If we allow that $[a, a] \in N$ for any $a \in L$, it then is easily seen that N forms a quotient ideal in L.

By the result of Y. Mayeta, we can find the congruence relation ξ such that N is equal to the set of all quotients which are nullized by the ξ.

Evidently $\alpha \equiv \beta \ (\xi)$ for α, $\beta \in$ any non-modular subset. This congruence relation ξ is called *modularlized*.

Let L/ξ be the quotient lattice of L by ξ, i.e., the set of all congruence classes by ξ. Then L/ξ is clarely a modular lattice, if one defines $\overline{x \cup y} = \bar{x} \cup \bar{y}$, $\overline{x \cap y} = \bar{x} \cap \bar{y}$, where $\bar{x}$ is the congruence class containing x.

Now we can prove the following.

LEMMA 4. *Let L be non-modular lattice in which any bounded chain is finite, and ξ the modularlized congruence relation of L. Then there exists a complement congruence relation ξ' of ξ.*

PROOF. Let S_1 be the set of projective classes having at least one prime quotient in $[a \cap b, a \cup b]$ for some $a \equiv b \ (\xi)$ and P the set of all projective classes of prime quotient on L. Set $S_2 = P - S_1$. And we define a new congruence relation ξ' as following: $x \equiv y \ (\xi')$ if and only if either $x = y$ or any prime quotient in $[x \cap y, x \cup y] \in$ some P_α in S_2, where P_α is a projective class of prime quotients of L. If any prime quotient in $[x \cap y, x \cup y] \in$ some P_α in S_2, then we can see: the quotient $[x \cap y, x \cup y]$ is a modular sublattice of L. For, if there exists a non-modular subset M in $[x \cap y, x \cup y]$, then we can find a projective class P_α in S_2 having a prime quotient $[\alpha, \beta]$ for α, $\beta \in M$. But on the other hand, since $\alpha \equiv \beta \ (\xi)$, we have $P_\alpha \in S_1$, which is unreasonable. Hence the quotient $[x \cap y, x \cup y]$ is a modular sublattice of L. Therefore we see that in the modular sublattice $[x \cap y,$

$x \cup y]$, lemma 1 and 2 are satisfied. Thus by a similar way as was done in the proof of lemma 3, we are easily seen that ξ' is a congruence relation, moreover $\xi \cap \xi' = O$ and $\xi \cup \xi' = I$.

It is easily seen that if a congruence relation θ in L satisfies $\alpha \equiv \beta \ (\theta)$ for all $\alpha, \ \beta \in$ any non-modular subset in L, then $\theta \geqq \xi$.

And we can prove:

LEMMA 5. *Let Ψ be the set of all congruence rlations θ such that $\alpha \equiv \beta \ (\theta)$ for all $\alpha, \ \beta \in$ any non-modular subsct in L, and Φ/ξ the set of all congruence relations defined on the quotient lattice L/ξ by ξ. Then Ψ is isomorphic to Φ/ξ.*

PROOF. For any $\theta \in \Psi$, clearly $\xi \leqq \theta$, i.e., $x \equiv y \ (\xi)$ implies $x \equiv y \ (\theta)$. If $a \equiv b \ (\xi)$ in L, then $\bar{a} = \bar{b}$ in L/ξ. Now we defined a new relation $\bar{\theta}$ on L/ξ as following: $\bar{a} \equiv \bar{b} \ (\theta)$ in L/ξ if and only if $a \equiv b \ (\theta)$ for some $a \in \bar{a}$ and $b \in \bar{b}$. Then we are easily seen that $\bar{\theta}$ is a congruence relation on L/ξ. Set $\bar{\theta} = f(\theta)$. Then f is one-to-one. For, for $\theta, \ \eta \in \Psi \ (\theta \neq \eta)$, there is a pair $a, \ b$ in L such that cither.

(i) $a \equiv b \ (\theta)$ and $a \not\equiv b \ (\eta)$ i.e., $a \cap b \equiv a \cup b \ (\theta)$ and $a \cap b \not\equiv a \cup b \ (\eta)$

or (ii) $a \not\equiv b \ (\theta)$ and $a \equiv b \ (\eta)$ i.e., $a \cap b \not\equiv a \cup b \ (\theta)$ and $a \cap b \equiv a \cup b \ (\eta)$

For the case (i), since $a \cap b \not\equiv a \cup b \ (\eta)$ there exists a prime quotient $[x, \ y]$ in $[a \cap b, \ a \cup b]$ such that $x \not\equiv y \ (\eta)$, which means $\bar{x} \not\equiv \bar{y} \ (\bar{\eta})$. On the other hand, $x \equiv y \ (\theta)$, which means $\bar{x} \equiv \bar{y} \ (\bar{\theta})$.

Hence $\bar{\theta} \neq \bar{\eta}$ i.e., $f(\theta) \neq f(\eta)$. The order preservings of f and f^{-1} is almost trivial.

Thus by theorem 1 we know that Φ/ξ is to be a Boolean algebra because L/ξ is a modular lattice. Hence Ψ is also a sub-Boolean algebra with zero element ξ and unit element I in Φ.

Now we shall prove that any $\theta \in \Psi$ has its complement congruence relation θ' in Φ.

Let θ be any congruence relation in Ψ. There exists a complement congruence relation $\bar{\theta}'$ of $\bar{\theta}$ on Φ/ξ. So we can find η in Ψ such that $f(\eta)=\bar{\theta}'$, where $\bar{\theta}=f(\theta)$. Setting $\theta'=\eta\cap\xi'$, we can see: θ' is a complement of θ in Φ. In fact, we first prove $\theta\cap\theta'=O$ in Φ. It suffices to show that $x\not\equiv y$ $(\theta\cap\theta')$ for any prime quotient $[x, y]$. If $x\equiv y$ (ξ), then $x\not\equiv y$ (ξ') i.e., $x\not\equiv y$ (θ') which follows $x\not\equiv y$ $(\theta\cap\theta')$. If $x\not\equiv y$ (ξ), then we have the following two cases:

(i) $x\equiv y$ (θ) and (ii) $x\not\equiv y$ (θ)

For (i), we have $\bar{x}\equiv\bar{y}$ $(f(\theta))$. Since $\bar{x}\neq\bar{y}$ in L/ξ, we can see: $x\not\equiv y$ (η) Thus $x\not\equiv y$ $(\theta\cap\theta')$. For (ii), we have immediately $x\not\equiv y$ $(\theta\cap\theta')$. Next we prove $\theta\cup\theta'=I$. Since Φ is distributive we have

$$\theta\cup\theta'=(\theta\cup\eta)\cap(\theta\cup\xi').$$

For any a and $b\in L$, let $a\cap b=a_0<a_1<\cdots\cdots<a_m=a\cup b$ be a maximal chain connecting $a\cap b$ and $a\cup b$.

If $a_{i-1}\equiv a_i$ (θ), then $a_{i-1}\equiv a_i$ $((\theta\cup\eta)\cap(\theta\cup\xi))$. If $a_{i-1}\not\equiv a_i$ (θ) for some i, then $\bar{a}_{i-1}\not\equiv\bar{a}_i$ $(f(\theta))$ in L/ξ which follows $a_{i-1}\equiv a_i$ (η). While $a_{i-1}\not\equiv a_i$ (θ), we have $a_{i-1}\not\equiv a_i$ (ξ) i.e., $a_{i-1}\equiv a_i$ (ξ') because a_i covers a_{i-1}. Hence $a_{i-1}\equiv a_i$ $((\theta\cup\eta)\cap(\theta\cup\xi'))$ for any i. It follows $a\equiv b$ $((\theta\cup\eta)\cap(\theta\cup\xi'))$, which completes the proof.

Hence we have the following theorem

THEOREM 2. *Let L be non-modular lattice in which all bounded chains are finite, and Ψ the set of all congruence relations by which all elements of each non-modular subset are nullized. Then for any $\theta\in\Psi$, there exists a complement of θ in Φ, and moreover $\Psi\cong 2^{Q}$, where Q is the set of all projective classes of the quotient lattice L/ξ. Conversely, let Ψ' be the set of each complement of $\theta\in\Psi$. Then each $\eta\in\Phi-(\Psi\vee\Psi')$ has not its complement.*

Now we shall show the converse of the theorem. First of all, we give the following obvious lemma:

LEMMA 6. (i) *Let θ be a congruence relation on a lattice L, and S a sublattice of L, then the contraction $[\theta]$ of θ on S is also a congruence relation of S.*

(ii) *If θ' is a complement of θ on L, then $[\theta']$ is also a complement of the $[\theta]$ on S.*

Let η be any congruence relation of L in $\Phi-(\Psi \vee \Psi')$. Since L is non-modular lattice, there exists at least one sublattice S isomorphic to the five-element lattice of Fig. 1. Thus the contraction $[\eta]$ of η on S is a congruence relation of S. Clearly the $[\eta]$ is not trivial congruence relations (O or I) on S. Thus $[\eta]$ is either

(α) $a \equiv b \equiv z$ $([\eta])$, $x \equiv u$ $([\eta])$ and $a \not\equiv x$ $([\eta])$,

or (β) $u \equiv a \equiv b$ $([\eta])$, $x \equiv z$ $([\eta])$ and $x \not\equiv u$ $([\eta])$ on Fig. 1

Suppose there exists a complement η' of η on L. Then $[\eta']$ is a complement of $[\eta]$ on S. But $[\eta']$ is neither (α) nor (β).

THEOREM 3. *In non-modular lattice in which all bounded chains are finite, Φ is a Boolean algebra if and only if $\Phi = \Psi \vee \Psi'$.*

3. L-ideals and l-congruence relations

In this section we shall denote by L an l-group [1].

LEMMA 7. *Let L be an l-group which satisfies the chain condition* [1]. *Then in L any bounded chain is finite*

PROOF. Let Υ be a chain connecting a to b. By the theorem 21 (in [1], p. 236) L is commutative and a and b are expressed uniquely as a sum of integeral multiples of finite number of distinct primes:

$$a=m_1p_1+m_2p_2+\cdots\cdots+m_rp_r,\quad b=n_1p_1+n_2p_2+\cdots\cdots+n_sp_s.$$

Where $r \leqq s$ and $n_i-m_i \geqq 0$, p_i is a prime in L.

Hence the length of $\Upsilon \leqq$

$$(n_1-m_1)+(n_2-m_2)+\cdots\cdots+(n_r-m_r)+n_{r+1}+\cdots\cdots+n_s.$$

COROLLARY. *Let L be an l-group which satisfies the chain condition, and Φ the set of all congruence relations defined on L. Then Φ forms a Boolean algebra.*

A congruence relation θ of L is called an *l-congruence* relation if and only if $a \equiv b\ (\theta)$ and $c \equiv d\ (\theta)$ imply $a+c \equiv b+d\ (\theta)$. And by an l-ideal $\mathfrak{a}$ of L is meant a normal subgroup of L in which $a,\ b \in \mathfrak{a}$ and $a \cap b \leqq x \leqq a \cup b$ imply $x \in \mathfrak{a}$.

Clearly we can see:

$$\Psi = \{\theta \mid \theta \text{ is l-congruence relation}\} \cong \{\mathfrak{a} \mid \mathfrak{a} \text{ is l-ideal}\}$$

LEMMA 8. *Let L be an l-group which satisfis the chain condition, and Q the set of prime quotient whose form is $[np, (n+1)p]$ for any prime p and an integer n. Then P and Q are one-to-one, where P is the set of all projective classes of prime quotients of L.*

PROOF. It is well known that b covers a if and only if for $a = np+mq+\cdots+lr$, $b=(n+1)p+mq+\cdots+lr$ or $b = np+(m+1)q+\cdots+lr$ or $\cdots$or $b = np+mq+\cdots+(l+1)r$. where $n, m, \cdots, l$ are integers and $p, q, \cdots, l$ prims. And we are easily seen that $[np, (n+1)p] \sim [a, b]$ (projective) if and only if $a=np+s$ and $b=(n+1)p+s$ $(p \notin s)$. Therefore a projective class P_α of prime quotients containing $[np, (n+1)p]$ is the set of all prime quotients whose form are $[np+s, (n+1)p+s]$. Thus by the corresponding: P_α $(\ni [np, (n+1)p]) \longrightarrow [np, (n+1)p]$, P and Q are one-to-one.

Denote $[p] = \{[np, (n+1)p] \mid n=0, \pm1, \pm2, \cdots\}$ for some prime p, and $R = \{[p], [q], \cdots\cdots\}$ for all primes $p, q, \cdots$, in L.

Since l-ideal $\mathfrak{a}$ is a subgroup of L, it is easy that $p \in \mathfrak{a}$ implies $np \in \mathfrak{a}$ for $n=0, \pm1, \pm2, \cdots$. Thus all prime quotients in $[p]$ are nullized by the l-congrence relation corresponding $\mathfrak{a}$. Hence clearly we have $\Phi \cong 2^Q$ and $\Psi \cong 2^R$.

Hence we have the following theorem

THEOREM 4. *Let L be an l-group which satisfies the chain condition, and $\Phi = \{$all congruence relations$\}$, $\Psi = \{$all l-congruence relations$\}$. Then Ψ is a sub-Boolean algebra of the Boolean algebra Φ, and moreover $\Psi \cong 2^R$, where $R = \{$all primes in $L\}$.*

COROLLARY. *The set of all l-ideals in L is an atomic complete Boolean algebra.* (This corollary was already proved in [1] p. 236).

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