ON THE PRODUCT GROUP MANIFOLD

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Introduction.

The differential geometric properties on the parameter group manifolds, as well as the semi-simple group manifolds were studied by many authors. N. Horie gave the metric $g_{\alpha\beta}=\sum_{a} A^a{}_\alpha A^a{}_\beta$ on the parameter group manifolds and he studied some differential geomtric properties of them [1], and further, the present author had some works on the parameter group manifolds [3], [4], [5], [6].

The present paper will show a definition of a group manifold, named, the product group manifold, which contains the first and second parameter group manifolds and moreover has the characteristics of them, and investigate some properties on this manifol d.

In section 1, we shall constitute a product group manifold and compare with the usual parameter group manifolds. In section 2, we shall find the structure tensor of this manifold and contrast the almost product manifold introduced by K. Yano [2]. In section 3, we shall investigate the integrabilities of the parameter group manifolds in this manifold. In section 4, we shall define a suitable connection in this manifold and and investigate some properties of this manifold with connection.

1. The product group manifold.

Let us consider the continuous transformation group of dimension r. which is defined by the equations

$$'x^i=f^i(x^1,\cdots,x^n;\ a^1,\cdots,a^r)\quad (i=1,\cdots,n),$$

where a's are r parameters and x's independent variables, and take the followings as equations of combinations in the group of dimensien r;

$$a_3{}^\alpha=\varphi^\alpha(a_1,a_2)\qquad (\alpha=1,\cdots,r). \tag{1. 1}$$

Then, the group defined by (1. 1), when $a_1{}^\alpha$ and $a_2{}^\alpha$ are fixed and $a_2{}^\alpha$ and $a_1{}^\alpha$ are considered as the parameters, are called respectively the first and second parameter groups $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$, That is, $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$ are defined by the equations

$$a_3{}^\alpha=\varphi^\alpha(c_1,\ a_2)\quad (c_1\text{'s}=\text{constants}), \tag{1. 2}$$

$$a_3{}^\alpha=\varphi^\alpha(a_1,\ c_2)\quad (c_2\text{'s}=\text{constants}).$$

We define a manifold $\mathscr{G}$ of dimension $2r$ which admits the equations

(1. 3) $$'a^{\lambda}=\Phi^{\lambda}(a^{\bar{1}},\cdots,a^{\bar{r}};a^{1},\cdots,a^{r})\quad(\lambda=1,\cdots,r;\bar{1},\cdots,\bar{r}).$$

satisfying the followings

(1. 4) $$\Phi^{\alpha}(c^{\bar{1}},\cdots,c^{\bar{r}};\ a^{1},\cdots,a^{r})=\varphi^{\alpha}(\bar{c},a),$$
$$\Phi^{\bar{\alpha}}(a^{\bar{1}},\cdots,a^{\bar{r}};\ c^{1},\cdots,c^{r})=\varphi^{\bar{\alpha}}(\bar{a},c),$$
$$\Phi^{\alpha}(a^{\bar{1}},\cdots,a^{\bar{r}};\ c^{1},\cdots,c^{r})=0,$$
$$\Phi^{\bar{\alpha}}(c^{\bar{1}},\cdots,c^{\bar{r}};\ a^{1},\cdots,a^{r})=0.$$
$$(\kappa,\lambda,\mu,\cdots=1,\cdots,r;\bar{1},\cdots,\bar{r};\ \alpha,\beta,\gamma,\cdots=1,\cdots,r;\ \bar{\alpha},\bar{\beta},\bar{\gamma},\cdots=\bar{1},\cdots,\bar{r})$$

Then, we may aware that $\mathscr{G}^{(+)}$ and $\mathscr{G}^{(-)}$ are defined by Φ^{α} and $\Phi^{\bar{\alpha}}$ in the manifold $\mathscr{G}$.

In $\mathscr{G}$, the fundamental equations of $\mathscr{G}^{(+)}$ and $\mathscr{G}^{(-)}$ are represented respectively by

(1. 5) $$\frac{\partial\Phi^{\alpha}}{\partial a^{\beta}}=A_{b}^{\ \beta}('a^{\alpha})A^{b}_{\ \beta}(a),\quad\frac{\partial\Phi^{\alpha}}{\partial a^{\bar{\beta}}}=0,$$

(1. 6) $$\frac{\partial\Phi^{\bar{\alpha}}}{\partial a^{\bar{\beta}}}=A_{\bar{b}}^{\ \bar{\alpha}}('a^{\alpha})A^{\bar{b}}_{\ \beta}(a),\quad\frac{\partial\Phi^{\bar{\alpha}}}{\partial a^{\beta}}=0,$$
$$(a,b,c,\cdots=1,\cdots,r;\ \bar{a},\bar{b},\bar{c},\cdots=\bar{1},\cdots,\bar{r}),$$

where the matrices $||A_{b}^{\ \alpha}||$, $||A_{\bar{b}}^{\ \bar{\alpha}}||$ and $||A^{b}_{\ \alpha}||$, $||A^{\bar{b}}_{\ \bar{\alpha}}||$ are inverse each other respectively, i.e.,

(1. 7) $$A_{a}^{\ \alpha}A^{a}_{\ \beta}=A^{\alpha}_{\beta},\quad A_{a}^{\ \alpha}A^{b}_{\ \alpha}=\delta^{b}_{a},$$
$$A_{\bar{a}}^{\ \bar{\alpha}}A^{\bar{a}}_{\ \bar{\beta}}=A^{\bar{\alpha}}_{\bar{\beta}},\quad A_{\bar{a}}^{\ \bar{\alpha}}A^{\bar{b}}_{\ \bar{\alpha}}=\delta^{\bar{b}}_{\bar{a}},$$

A^{λ}_{μ} being unit tensor and δ Kronecker symbol.

It is well known that r-vectors $A_{a}^{\ \alpha}(a=1,\cdots,r)$ in $\mathscr{G}^{(+)}$ and r-vectors $A_{\bar{a}}^{\ \bar{\alpha}}$ $(\bar{a}=\bar{1},\cdots,\bar{r})$ in $\mathscr{G}^{(-)}$ are linearly independent respectively. Thus, we may construct $2r$ linearly indendent vectors $A_{m}^{\ \lambda}(l,m,n=1,\cdots,r;\bar{1},\cdots,\bar{r})$ in $\mathscr{G}$. In fact, especially, $2r$ vectors $(A_{b}^{\ \alpha},\ 0)$, $(0,\ A_{\bar{b}}^{\ \bar{\alpha}})$ are linearly independent in $\mathscr{G}$. And we take the inverse matrix $||A^{m}_{\ \lambda}||$ of $||A_{m}^{\ \lambda}||$, and then we have the relations

(1. 8) $$A_{m}^{\ \mu}A^{m}_{\ \lambda}=A^{\mu}_{\lambda},\quad A_{m}^{\ \lambda}A^{n}_{\ \lambda}=\delta^{n}_{m}.$$

From (1. 7), we have the followings;

(1. 9) $$A_a{}^{\bar\alpha}A^b{}_{\bar\alpha}=0,\ A_{\bar a}{}^{\alpha}A^{\bar b}{}_{\alpha}=0,\ A_{\bar a}{}^{\alpha}A^{\bar a}{}_{\beta}=0,\ A_a{}^{\bar\alpha}A^a{}_{\bar\beta}=0,$$

(1.10) $$A_a{}^{\alpha}A^{\bar b}{}_{\alpha}=-A_a{}^{\bar\alpha}A^{\bar b}{}_{\bar\alpha},\ A_{\bar a}{}^{\alpha}A^b{}_{\alpha}=-A_{\bar a}{}^{\bar\alpha}A^b{}_{\bar\alpha},$$
$$A_a{}^{\alpha}A^a{}_{\bar\beta}=-A_{\bar a}{}^{\alpha}A^{\bar a}{}_{\bar\beta},\ A_a{}^{\bar\alpha}A^a{}_{\beta}=-A_{\bar a}{}^{\bar\alpha}A^{\bar a}{}_{\beta}.$$

And we have from (1. 8) and (1. 9)

(1.11) $$A_b{}^{\alpha}A^b{}_{\beta}A^{\bar c}{}_{\alpha}=0,\quad A^b{}_{\alpha}A_b{}^{\beta}A_{\bar c}{}^{\alpha}=0,$$
$$A_{\bar b}{}^{\bar\alpha}A^{\bar b}{}_{\beta}A^c{}_{\bar\alpha}=0,\quad A^{\bar b}{}_{\bar\alpha}A_{\bar b}{}^{\beta}A_c{}^{\bar\alpha}=0.$$

Let us assume that the quantities $A^a{}_\lambda$ and $A_a{}^\lambda$ are functions depended only upon a's and $A^{\bar a}{}_\lambda$ and $A_{\bar a}{}^\lambda$ only upon $\bar a$'s.

A $2r$-dimensional manifold $\mathscr{G}$ admiting the equations (1. 3) and the linearly independent vector fields $A_m{}^\lambda$ is called by a *product group manifold.*

In $\mathscr{G}^{(+)}$ and $\mathscr{G}^{(-)}$, they satisfy Maurer-Cartan equations:

(1.12) $$(\partial_\gamma A^a{}_\beta-\partial_\beta A^a{}_\gamma)\ A^c{}_\gamma A^b{}_\beta\ =\ C^a_{cb},$$
$$(\partial_{\bar\gamma} A^{\bar a}{}_{\bar\beta}-\partial_{\bar\beta} A^{\bar a}{}_{\bar\gamma})\ A^{\bar c}{}_{\bar\gamma} A^{\bar b}{}_{\bar\beta}\ =\ C^{\bar a}_{\bar c\bar b},$$

where C^a_{cb} and $C^{\bar a}_{\bar c\bar b}$ are constants of structure in $\mathscr{G}^{(+)}$ and $\mathscr{G}^{(-)}$ respectively, and ∂_λ denotes the partial differentiation with respect to a^λ.

We now define the quantities Ω^l_{nm} by

(1.13) $$\Omega^l_{nm}\ =\ (\partial_\mu A^l{}_\lambda-\partial_\lambda A^l{}_\mu)A_n{}^\mu A_m{}^\lambda.$$

Then we have

$$\Omega^a_{cb}\ =\ (\partial_\gamma A^a{}_\beta-\partial_\beta A^a{}_\gamma)A_c{}^\gamma A_b{}^\beta\ +\ (-\partial_\beta A^a{}_{\bar\gamma})A_c{}^{\bar\gamma}A_b{}^\beta\ +\ (\partial_\gamma A^a{}_{\bar\beta})A_c{}^\gamma A_b{}^{\bar\beta}.$$

Multiplying by $A^c{}_\varepsilon A^b{}_\delta$ and summing for b and c, since $(\partial_\varepsilon A^a{}_{\bar\gamma})A_{\bar c}{}^{\bar\gamma}=-(\partial_\varepsilon A^a{}_\gamma)A_{\bar c}{}^\gamma$ are satisfied by (1.10) and our assumption, it is reducible by means of (1. 9) and (1.10) into

$$\Omega^a_{cb}A^c{}_\varepsilon A^b{}_\delta\ =\ (\partial_\varepsilon A^a{}_\delta-\partial_\delta A^a{}_\varepsilon),$$

and thus,

$$\Omega^a_{cb}\ =\ (\partial_\gamma A^a{}_\beta-\partial_\beta A^a{}_\gamma)A_c{}^\gamma A_b{}^\beta,$$

or

(1.14) $$\Omega^a_{cb}=C^a_{cb}.$$

Similarly, we have

(1.15) $$\Omega^{\bar a}_{\bar c\bar b}=C^{\bar a}_{\bar c\bar b}.$$

And, from (1.13), we have

$$\Omega^{a}_{\bar{c}b} = (\partial_\gamma A^a{}_\beta - \partial_\beta A^a{}_\gamma) A_{\bar{c}}{}^\gamma A_b{}^\beta + (\partial_\gamma A^a{}_{\bar{\beta}}) A_{\bar{c}}{}^\gamma A_b{}^{\bar{\beta}} - (\partial_\beta A^a{}_{\bar{\gamma}}) A_{\bar{c}}{}^{\bar{\gamma}} A_b{}^\beta.$$

Using of (1.10) and multiplying $A^b{}_\delta$ and summing for b, we have by means of (1. 9)

$$\Omega^{a}_{\bar{c}b} A^b{}_\delta = (\partial_\gamma A^a{}_\delta) A_{\bar{c}}{}^\beta,$$

and thus

(1.16) $$\Omega^{a}_{\bar{c}b} = (\partial_\gamma A^a{}_\beta) A_{\bar{c}}{}^\gamma A_b{}^\beta = -\Omega^{a}_{b\bar{c}}.$$

Similarly, we have

(1.17) $$\Omega^{\bar{a}}_{c\bar{b}} = (\partial_\gamma A^{\bar{a}}{}_\beta) A_c{}^\gamma A_{\bar{b}}{}^\beta = -\Omega^{\bar{a}}_{\bar{b}c}.$$

From (1.13), we have

$$\Omega^{a}_{\bar{c}\bar{b}} = (\partial_\gamma A^a{}_\beta - \partial_\beta A^a{}_\gamma) A_{\bar{c}}{}^\gamma A_{\bar{b}}{}^\beta + (\partial_\gamma A^a{}_{\bar{\beta}}) A_{\bar{c}}{}^\gamma A_{\bar{b}}{}^{\bar{\beta}} - (\partial_\beta A^a{}_{\bar{\gamma}}) A_{\bar{c}}{}^{\bar{\gamma}} A_{\bar{b}}{}^\beta.$$

Then it is reducible means of (1.10) into

(1.18) $$\Omega^{a}_{\bar{c}\bar{b}} = 0.$$

Similarly, we have

(1.19) $$\Omega^{\bar{a}}_{cb} = 0.$$

Thus we know that *the quantities Ω^l_{mn} are charaterized by formulas from* (1.14) *to* (1.19).

2. The structure of the product group manifold.

From the construction of the product group manifold $\mathcal{G}$, we may notice that the parmeter group manifolds $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$ are immersed in the manifold $\mathcal{G}$. Let us consider $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$ as immersed manifolds of the both dimension r in $\mathcal{G}$, then $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$ are constituted by the linearly independent vectors $A_a{}^\lambda (a=1, \cdots, r)$ and $A_{\bar{a}}{}^\lambda (\bar{a}=\bar{1}, \cdots, \bar{r})$ in $\mathcal{G}$.

If we put

(2. 1) $$A_a{}^\lambda A^a{}_\mu = E^\lambda_\mu, \quad A_{\bar{a}}{}^\lambda A^{\bar{a}}{}_\mu = \bar{E}^\lambda_\mu,$$

then, from (1. 9) and (1.10), they are charaterized by

(2. 2) $$E^\alpha_\beta = A^\alpha_\beta, \ E^{\bar{\alpha}}_{\bar{\beta}} = 0, \ \bar{E}^\alpha_\beta = 0, \ \bar{E}^{\bar{\alpha}}_{\bar{\beta}} = A^{\bar{\alpha}}_{\bar{\beta}},$$

and

(2. 3) $$E^\alpha_{\bar{\beta}} + \bar{E}^\alpha_{\bar{\beta}} = 0, \ E^{\bar{\alpha}}_\beta + \bar{E}^{\bar{\alpha}}_\beta = 0.$$

And from (2. 2) and (2. 3), we have

(2. 4) $$E^{\lambda}_{\mu} + \bar{E}^{\lambda}_{\mu} = A^{\lambda}_{\mu}$$

Since $E^{\alpha}_{\beta}=A_a{}^{\alpha}A^a{}_{\beta}$, $E^{\bar{\alpha}}{}_{\beta}=A_a{}^{\bar{\alpha}}A^a{}_{\beta}$, $\bar{E}^{\alpha}_{\beta}=A_{\bar{a}}{}^{\alpha}A^{\bar{a}}{}_{\beta}$, and $\bar{E}^{\bar{\alpha}}_{\beta}=A_{\bar{a}}{}^{\bar{\alpha}}A^{\bar{a}}{}_{\beta}$, we can aware that E^{α}_{β} and $E^{\bar{\alpha}}_{\beta}$ are depend only upon a's and $\bar{E}^{\alpha}_{\beta}$, $\bar{E}^{\bar{\alpha}}_{\beta}$ only upon $\bar{a}$'s, and consequently, from (2. 3), we can see that the tensors E^{λ}_{μ} and $\bar{E}^{\lambda}_{\mu}$ are constant.

From (2. 1), we have

(2. 5) $$E^{\kappa}_{\lambda}\, E^{\lambda}_{\mu}=E^{\kappa}_{\mu},\ E^{\kappa}_{\lambda}\, \bar{E}^{\lambda}_{\mu}=0,\ \bar{E}^{\kappa}_{\lambda}\, E^{\lambda}_{\mu}=0,\ \bar{E}^{\kappa}_{\lambda}\, \bar{E}^{\lambda}_{\mu}=\bar{E}^{\kappa}_{\mu}.$$

We now define the tensor F^{κ}_{λ} by

(2. 6) $$F^{\kappa}_{\lambda} = E^{\kappa}_{\lambda} - \bar{E}^{\kappa}_{\lambda},$$

then (2. 4) and (2. 6) give

$$E^{\kappa}_{\lambda} = \frac{1}{2}(A^{\kappa}_{\lambda} + F^{\kappa}_{\lambda}),\ \bar{E}^{\kappa}_{\lambda} = \frac{1}{2}(A^{\kappa}_{\lambda} - F^{\kappa}_{\lambda}),$$

and, since E^{κ}_{λ} and $\bar{E}^{\kappa}_{\lambda}$ are constant tensors, we know that the tensor F^{κ}_{λ} is constant.

Using of (2. 5), we can easily see that

(2. 7) $$F^{\kappa}_{\lambda}\, F^{\lambda}_{\mu} = A^{\kappa}_{\mu}.$$

Hence, we have the following:

THEOREM 1. *The product group manifold is an almost product manifold with the constanst structure tensor* F^{κ}_{λ} [2].

3. Integrabilities.

Let us take the paffian derivative ∂_l with respect to $A_l{}^{\lambda}$, i.e,

(3. 1) $$\partial_l=A_l{}^{\lambda}\partial_{\lambda}=A_l{}^{\lambda}\frac{\partial}{\partial a^{\lambda}},$$

where a^{λ} are local coordinates. We call the set of $A_l{}^{\lambda}$ the *fundamental frame of* $\mathcal{G}$. Then (1.10) are reducible into

(3. 2) $$\Omega^{l}_{nm} = (\partial_n A_m{}^{\lambda}-\partial_m A_n{}^{\lambda})A^{l}{}_{\lambda}.$$

If we effect the transformation of the fundamental frame;

(3. 3) $$A_{l'}{}^{\lambda}=A^{l}_{l'}A_l{}^{\lambda},$$

then we can easily see that Ω^{l}_{nm} undergoes the transformation

(3. 4) $$\Omega^{l'}_{n'm'} = (\partial_{n'}A^{l}_{m'}-\partial_{m'}A^{l}_{n'})A^{l'}_{l} + A^{l'}_{l}A^{n}_{n'}A^{m}_{m'}\Omega^{l}_{nm},$$

where the matrix $||A^{l'}_{l}||$ is inverse of $||A^{l}_{l'}||$.

Since we may consider a fundamental frame whose first r-vectors are in the manifold $\mathcal{G}^{(+)}$ and whose second r-vectors are in the manifold $\mathcal{G}^{(-)}$, the transformation (3. 3) of the fundamental frame must split into

$$A_{a'}{}^{\lambda}=A^{a}_{a'}A_{a}{}^{\lambda},\quad A_{\bar{a}'}{}^{\lambda}=A^{\bar{a}}_{\bar{a}'}A_{\bar{a}}{}^{\lambda}, \tag{3. 5}$$

that is,

$$||A^{l}_{l'}|| = \begin{pmatrix} A^{a}_{a'} & 0 \\ 0 & A^{\bar{a}}_{\bar{a}'} \end{pmatrix}. \tag{3. 6}$$

Hence, we have from (1.18) and (1.19)

$$\Omega^{a'}_{\bar{b}'\bar{c}'}=0,\quad \Omega^{\bar{a}'}_{b'c'}=0. \tag{3. 7}$$

An arbitrary contravariant vector da^{λ} in the manifold $\mathcal{G}$ at $a \in \mathcal{G}$ can be represented by the form, since $A_{a}{}^{\lambda}$, $A_{\bar{a}}{}^{\lambda}$ are linearly independent in $\mathcal{G}$,

$$da^{\lambda}=A_{a}{}^{\lambda}(da)^{a}+A_{\bar{a}}{}^{\lambda}(da)^{\bar{a}}. \tag{3. 8}$$

Contracting $A^{b}{}_{\lambda}$ and $A^{\bar{b}}{}_{\lambda}$ with respect to λ, then we have from (1. 8)

$$(da)^{b}=A^{b}{}_{\lambda}da^{\lambda},\qquad (da)^{\bar{b}}=A^{\bar{b}}{}_{\lambda}da^{\lambda}. \tag{3. 9}$$

Thus, if we consider $\mathcal{G}$ as the vector space, the manifold $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$ are defined respectively by

$$\begin{aligned} (da)^{\bar{a}}&=A^{\bar{a}}{}_{\lambda}da^{\lambda}=0, \\ (da)^{a}&=A^{a}{}_{\lambda}da^{\lambda}=0. \end{aligned} \tag{3.10}$$

Then the conditions of the complete integrability of $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$ are that

$$(\partial_{\mu}A^{\bar{a}}{}_{\lambda}-\partial_{\lambda}A^{\bar{a}}{}_{\mu})da^{\mu}\wedge da^{\lambda}=0,$$

$$(\partial_{\mu}A^{a}{}_{\lambda}-\partial_{\lambda}A^{a}{}_{\mu})da^{\mu}\wedge da^{\lambda}=0$$

are held for any da^{λ} satisfying (3. 9), and these are equivalent to

$$A_{c}{}^{\mu}A_{b}{}^{\lambda}(\partial_{\mu}A^{\bar{a}}{}_{\lambda}-\partial_{\lambda}A^{\bar{a}}{}_{\mu})=0,$$

$$A_{\bar{c}}{}^{\mu}A_{\bar{b}}{}^{\lambda}(\partial_{\mu}A^{a}{}_{\lambda}-\partial_{\lambda}A^{a}{}_{\mu})=0,$$

i.e.,

$$\Omega^{\bar{a}}_{cb}=0,\quad \Omega^{a}_{\bar{c}\bar{b}}=0.$$

These are satisfied identically in the manifold $\mathcal{G}$, and we have form (3. 7) the followings:

THEOREM 2. *In the product group manifold $\mathcal{G}$, the parameter group manifolds $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$ are always conpletely intgrable, and their* *integrabilities of $\mathscr{G}^{(+)}$ and $\mathscr{G}^{(-)}$ are preserved by the transformation of the fundamental frame.*

4. Connections and path.

If we put

(4. 1) $$L^{\lambda}_{\nu\mu} = A_m{}^{\lambda}(\partial_\nu A^m{}_\mu) = -(\partial_\nu A_m{}^{\lambda})A^m{}_\mu,$$

then we can easily see that $L^{\lambda}_{\nu\mu}$ is a connection in $\mathscr{G}$.

Let us take the quantities L^l_{nm}, named, *a connection with respect to the fundamental frame,* such that

(4. 2) $$L^l_{nm} = (\partial_n A_m{}^{\lambda})A^l{}_\lambda + A_n{}^{\nu}A_m{}^{\mu}A^l{}_\lambda L^{\lambda}_{\nu\mu}.$$

Then we see that

(4. 3) $$L^l_{nm}=0.$$

And if we take another connection $\Gamma^{\lambda}_{\nu\mu}$ in $\mathscr{G}$;

(4. 4) $$\Gamma^{\lambda}_{\nu\mu}=\frac{1}{2}(L^{\lambda}_{\nu\mu} + L^{\lambda}_{\mu\nu}),$$

then the connection Γ^l_{nm} with respect to $A_m{}^{\lambda}$, i.e.,

(4. 5) $$\Gamma^l_{nm} = (\partial_n A_m{}^{\lambda})A^l{}_\lambda + A_n{}^{\nu}A_m{}^{\mu}A^l{}_\lambda L^{\lambda}_{\nu\mu}$$

are, from (4. 1) and (3. 2), reducible into

(4. 6) $$\Gamma^l_{nm} = \frac{1}{2}\Omega^l_{nm}.$$

Next, if we take $\Omega^{\lambda}_{\nu\mu}$ as the other connection such that

(4. 7) $$\Omega^{\lambda}_{\nu\mu} = 2L^{\lambda}_{\nu\mu} - L^{\lambda}_{\mu\nu},$$

it is reducible by means of (4. 1) into

$$\Omega^{\lambda}_{\nu\mu} = (\partial_\nu A^l{}_\mu)A_l{}^{\lambda} + (\partial_\tau A^l{}_\sigma-\partial_\sigma A^l{}_\tau)A_n{}^{\tau}A_m{}^{\sigma}A^n{}_\nu A^m{}_\mu A_l{}^{\lambda}$$

and thus, we have from (1.13)

(4. 8) $$\Omega^{\lambda}_{\nu\mu} = (\partial_\nu A^l{}_\mu)A_l{}^{\lambda} + \Omega^l_{nm}A^n{}_\nu A^m{}_\mu A_l{}^{\lambda}.$$

Using of (4. 1), it reduces into

(4. 9) $$\Omega^l_{mn} = (\partial_n A_m{}^{\lambda})A^l{}_\lambda + \Omega^{\lambda}_{\nu\mu}A_n{}^{\nu}A_m{}^{\mu}A^l{}_\lambda,$$

and hence, the quantites Ω^l_{nm} defined by (1.13) are the components of a connection of $\Omega^{\lambda}_{\nu\mu}$ with respect to $A_l{}^{\lambda}$.

Now, let us find the completely intgrable condition of (4. 8). Acting (4. 8) by the operation ∂_ρ and substituting (4. 8) in it, then we have

$$\partial_\rho\partial_\nu A^l{}_\mu = (\partial_\rho\Omega^\lambda_{\nu\mu} + \Omega\rho^\lambda_\sigma\ \Omega^\sigma_{\nu\mu})A^l{}_\lambda - (\partial_t\Omega^l_{sr} + \Omega^l_{tm}\ \Omega^m_{sr})A^t{}_\rho A^s{}_\nu A^r{}_\mu$$
$$- \Omega^\lambda_{\mu\nu}\ \Omega^l_{sr}A^s{}_\rho A^r{}_\lambda - \Omega^\lambda_{\rho\mu}\ \Omega^l_{sm}A^m{}_\nu A^r{}_\lambda - \Omega^l_{sr}(\partial_\rho A^s{}_\nu)A^r{}_\mu.$$

From the completely integrable condition $\partial_\rho\partial_\nu A^l{}_\mu = \partial_\nu\partial_\rho A^l{}_\mu$, it must satisfy the form

$$\Omega_{\rho\nu\mu}{}^\lambda = (\Omega_{tsr}{}^l - \Omega^m_{ts}\ \Omega^l_{rm})A^t{}_\rho A^s{}_\nu A^r{}_\mu A_l^\lambda, \tag{4.10}$$

where

$$\Omega_{\rho\nu\mu}{}^\lambda = \partial_\rho\Omega^\lambda_{\nu\mu} - \partial_\nu\Omega^\lambda_{\rho\mu} + \Omega^\lambda_{\rho\sigma}\ \Omega^\sigma_{\nu\mu} - \Omega^\lambda_{\nu\sigma}\ \Omega^\sigma_{\rho\mu}, \tag{4.11}$$

$$\Omega_{tsr}{}^l = \partial_t\Omega^l_{sr} - \partial_s\Omega^l_{tr} + \Omega^l_{tm}\ \Omega^m_{sr} - \Omega^l_{sm}\ \Omega^m_{tr} \tag{4.12}$$

We are well known that it is called for the manifold $\mathscr{G}$ to be flat, if it satisfies $\Omega_{\rho\nu\mu}{}^\lambda = 0$, and hence we have the following:

THEOREM 3. *For the product group manifold $\mathscr{G}$ to be flat, it is necessary and sufficient that*

$$\Omega_{tsr}{}^l = \Omega^m_{ts}\ \Omega^l_{rm} \tag{4.13}$$

If we take $l=a$, $r=b$, $s=c$ and $t=d$ in (4.12), in consequece of (1.19), (4.12), (4.13) and Jacobi identities

$$C^e_{bc}\ C^a_{de} + C^e_{cd}\ C^a_{be} + C^e_{db}\ C^a_{ce} = 0, \tag{4.14}$$

we have

$$C^e_{cu}C^a_{\flat e} = 0. \tag{4.15}$$

And similarly, using of (1.18), we have

$$C^{\bar e}_{\bar c\bar r}C^{\bar a}_{\bar b\bar e} = 0. \tag{4.16}$$

Hence we have the following:

COROLLARY. *If the product group manifold $\mathscr{G}$ is flat, then the parameter group manifolds $\mathscr{G}^{(+)}$ and $\mathscr{G}^{(-)}$ are characterized by the forms* (4.15) *and* (4.16) *respectively.*

The curve $a^\lambda(t)$ in the manifold $\mathscr{G}$ is called by an *one parameter product sub-group*, if it is the solutions of the differential equation

$$\frac{da^\lambda}{dt} = e^l A_l{}^\lambda(a), \tag{4.17}$$

where e's are constants and one of them at least is not zero.

We shall find the conditions that the tangents to the one parameter product sub-groups are always contained in the manifolds $\mathscr{G}^{(+)}$ and $\mathscr{G}^{(-)}$ for any initial point. Then we may assert our condition for $\mathcal{G}^{(+)}$ that the equation

$$A^{\bar{a}}{}_{\lambda}\frac{da^{\lambda}}{dt}=0$$

is satisfied along the equation (4.17) for any initial point. Thus, substituting (4.17) in it, we have

$$e^{l}A^{\bar{a}}{}_{\lambda}A_{l}{}^{\lambda}=0,$$

or

$$e^{\bar{a}}=0. \tag{4.18}$$

And similarly we have

$$e^{a}=0. \tag{4.19}$$

as the condition for the tangent to (4.17) to be contained in $\mathcal{G}^{(-)}$. Hence, we have the followings:

THEOREM 4. *In the product group manifold $\mathcal{G}$, for the tangent to the one parameter product sub-group to be always contained in the parameter group manifolds $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$, it is necessary and sufficient that it holds* (4.18) *and* (4.19) *respectively, and consequently that the one parameter product sub-group is one parameter sub-groups in $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$ respectively.*

For the connection $\Omega^{\kappa}_{\mu\lambda}$, the equation of path in $\mathcal{G}$ is represented by

$$\frac{d^2a^{\kappa}}{dt^2}+\Omega^{\kappa}_{\nu\mu}\frac{da^{\nu}}{dt}\frac{da^{\mu}}{dt}=\varphi\frac{da^{\kappa}}{dt}, \tag{4.20}$$

where t is any parameter and φ is a function of t.

We shall find the condition that the path (4.20) and the one parameter product sub-group are coincide each other.

Differentiating (4.17) by parameter t, we have

$$\frac{d^2a^{\kappa}}{dt^2}=-e^{n}e^{m}A_{n}{}^{\mu}A_{m}{}^{\lambda}L^{\kappa}_{\mu\lambda},$$

and consequently, (4.20) is reducible by means of this and (4.17) into

$$-e^{n}e^{m}A_{n}{}^{\mu}A_{m}{}^{\lambda}L^{\kappa}_{\mu\lambda}+\Omega^{\kappa}_{\mu\lambda}e^{n}e^{m}A_{n}{}^{\mu}A_{m}{}^{\lambda}=\varphi e^{r}A_{r}{}^{\kappa}.$$

From (4. 7), (4. 1) and (3. 2), we obtain

$$e^{n}e^{m}\,\Omega^{r}_{nm}\,A_{r}{}^{\kappa}=\varphi e^{r}A_{r}{}^{\kappa}.$$

Since Ω^{r}_{nm} is skew symmetric with respect to m and n, we have

$$\varphi e^{r}=0.$$

Since one of e's at least is not zero, we may assert that

(4.21) $\varphi=0$.

Hence we have the following:

THEOREM 5. *In the procuct group manifold $\mathcal{G}$, for the path with respect to connection $\Omega_{\nu\mu}^{\kappa}$ and the one parameter product sub-group to be coincide each other, it is necessary and sufficient that the given parameter is an affine parameter* [7].

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