ON ORDER-CONVERGENCE OF PARTIALLY ORDERED GROUPS

By Tae Ho Choe

1 Introduction. A partially ordered group X is (i) a partially ordered (ii) a group, in which (iii) the inclusion relation is invariant under all group-translations: $x \to a+x+b$ for any $a, x, b \epsilon X$ [2]. Let $\{f(d), d \epsilon D\}$ be a net on the directed set D with values in a partially ordered group X. In a partially ordered set, G. Birkhoff has defined the concept "o-convergence" by making use of a net (or directed set) which has been studied by Frink, McShane and Wolk [1].

In this paper, we shall apply "o-convergence" to the partially ordered group X to introduce the order topology to X. First of all, we shall find the sufficient conditions for a net f to order-converge to an element in a partially ordered set. Making use of these sufficient conditions on the σ-lattice, we shall find a necessary and sufficient condition that an element of X be an isolated point of X under it's order topology. And we shall show that on a complete l-group, the necessary and sufficient condition for X to be discrete under it's order topology is that X must have a chain condition. Finally, we shall give some other properties of the partially ordered group X in order that an element of X be a limiting point of o-convergence for some net f.

2 Preliminaries. We here recollect some terms and notations [1]. Let X be a set partially ordered by a relation $\leqq$. If S is a subset of X, we write

$$S^* = \{x \epsilon X \mid x \geqq a \text{ for all } a \epsilon S\}, \quad S^+ = \{x \epsilon X \mid x \leqq a \text{ for all } a \epsilon S\}.$$

Let us call a subset S of X *up-directed* (*down-directed*) if and only if for all $x \epsilon S$, and $y \epsilon S$ there exists $z \epsilon S$ such that $z \geqq x, z \geqq y$ ($z \leqq x, z \leqq y$). For nets $\{f(\alpha), \alpha \epsilon D\}$ our terminology and notation are those of Kelley [4].

We give the Birkhoff-Frink-McShane definition of o-convergence.

DEFINITION. If $\{f(\alpha), \alpha \epsilon D\}$ is a net in a partially ordered set X, we say that f *o-converges to* y (and write $y = o\text{-}\lim f$) if and only if there exist subsets M and N of X such that

(i) M is up-directed and N is down-directed.

(ii) $y = \text{l.u.b. } M = \text{g.l.b. } N$,

(iii) for each $m \epsilon M$ and $n \epsilon N$, there exists $\beta \epsilon D$ such that $m \leqq f(\alpha) \leqq n$ for all $\alpha \geqq \beta$.

One verifies easily the following formulas

(a) If $\{f(\alpha), \alpha \epsilon D\}$ is a net in X and $f(\alpha) = a$ for all α, then $a = o\text{-lim } f$.

(b) If $a = o\text{-lim } f$ and $b = o\text{-lim } f$, then $a = b$.

(c) If $a = o\text{-lim } f$ and $\{g(\sigma), \sigma \epsilon D'\}$ is a cofinal subnet of f, then $a = o\text{-lim } f$.

The following lemma will be of some use to us.

LEMMA 1. *Let X be a partially ordered set and a an element of X. If there exists a chain C such that $a \notin C$ and $a = l.u.b.\ C$ (or $= g.l.b.\ C$), then $a = o\text{-lim } f$ for some net f in $X - \{a\}$.*

PROOF. If $a = \text{l.u.b. } C$ and $a \notin C$, then C is an infinite chain. It is easy to find a directed partially ordered set D such that C is isotone image of D, i.e., there is an isotone f: $\alpha \leqq \beta$ in D implies $f(\alpha) \leqq f(\beta)$ in C, and $C = \{f(\alpha), \alpha \epsilon D\}$. If we take $M = C, N = \{a\}$, then (i) and (ii) are satisfied. And for each $b \epsilon M$ and $a \epsilon N$, there exists $\alpha \epsilon D$ such that $b = f(\alpha)$ and $b \leqq f(\beta) \leqq a$ for all $\beta \geqq \alpha$. Hence $a = o\text{-lim } f$. And $C = \{f(\alpha), \alpha \epsilon D\} \subseteq X - \{a\}$.

As usual, we define a subset S of a partially ordered set X to be *closed* under the order topology, if and only if $\{f(\alpha), \alpha \epsilon D\}$ is a net in S and $a = o\text{-lim } f$ imply $a \epsilon S$.

Then, as well known, under the order topology partially ordered set is a Hausdorff space, and any closed interval is closed under the order toplogy [2].

As a corollary of lemma 1 we have the following.

COROLLARY. *Let X be a partially ordered set and a an isolated point of X under it's order topology. Then there exist two subsets $P = \{x \epsilon X \mid x$ covers $a\}$, $Q = \{x \epsilon X \mid x$ is covered by $a\}$ of X such that every element over a (not a) is over an element $x \epsilon P$ and every element under a (not a) is under an element $y \epsilon Q$.*

Let $\{f(\alpha), \alpha \epsilon D\}$ be a net in X. If the directed set D is countable total ordered, then we call it a *ordinary* net, and denoted by $\{f(m), m \epsilon D\}$. And we can introduce the *ordinary* order topology in a partially ordered set X by making use of the above ordinary net $\{f(m), m\epsilon D\}$.

We shall use the above corollary to prove the following.

THEOREM 1. *Let X be a σ-lattice. The element a is an isolated point of X under the ordinary order topology if and only if there exist two subsets $P=\{x\epsilon X \mid x$ covers $a\}$, $Q=\{x\epsilon X \mid x$ is covered by $a\}$ of X such that every element of any chain in $\{a\}^{*}-\{a\}$ is over an element $x\epsilon P$, and every element of any chain in $\{a\}^{+}-\{a\}$ is under an element $x\epsilon Q$,*

PROOF. By the above corollary, the necessity is obvious. To prove the converse, we shall show that the subset $X-\{a\}$ is closed. Suppose $X-\{a\}$ is not closed: there exists a net $\{f(m), m\epsilon D\}$ in $X-\{a\}$ and $a=o\text{-lim} f$, i.e., there exist subsets M and N of X which satisfy (i), (ii) and (iii). Setting $u_n=\text{l.u.b.}\{f(m) \mid m\geqq n\}$, $v_n=\text{g.l.b.}\{f(m) \mid m\geqq n\}$, we see that $v_n\leqq f(n)\leqq u_n$ for all $n\epsilon D$, and $a=\text{g.l.b.}\{u_n \mid n\epsilon D\}=\text{l.u.b.}\{v_n \mid n\epsilon D\}$. In fact, by (ii) and (iii) a is a lower bound of $\{u_n \mid n\epsilon D\}$. Suppose $b\leqq v_n$ for all $n\epsilon D$. By (iii), for every $x\epsilon N$ there exists $n\epsilon D$ such that $f(m)\leqq x$ for all $m\geqq n$. It follows $u_n\leqq x$. Therefore b is a lower bound of N. By (ii) we have $b\leqq a$ Hence $a=\text{g.l.b.}\{u_n \mid n\epsilon D\}$, and dually. If $a=v_n$ for some n, then we have $a\neq u_n$ for any $n\epsilon D$. In fact, suppose $a=u_m$ for some $m\epsilon D$. Then we have two cases: (1) $m\geqq n$, we have $a=v_h$ for all $h\geqq n$ since $a=\text{l.u.b.}\{v_n\}$. Hence $a=f(m)$ which is contrary. (2) $m<n$, we similarly have $a=f(n)$ which is also contrary. In a similar way, if $a=u_n$ for some n, then we have $a\neq v_n$ for all $n\epsilon D$. For both cases, we have either $a\neq v_n$ for all $n\epsilon D$ or $a\neq u_n$ for all $n\epsilon D$. Say $a\neq u_n$ for all $n\epsilon D$. Since $\{u_n\}$ is a chain not containing a, by hypothesis there exists the element x of P which covers a and satisfies $x\leqq u_n$ for all $n\epsilon D$. But this contradicts to $a=\text{g.l.b.}\{u_n\}$.

3 Order topology in po-group. Let X be a partially orderd group. For a net $\{f(\alpha), \alpha\epsilon D\}$ and $y\epsilon X$, we define a new net $\{f_y(\alpha), \alpha\epsilon D\}$ such that $f_y(\alpha)=f(\alpha)+y$ for each $\alpha\epsilon D$. Then we have the following theorem

THEOREM 2. *Let X be a partially ordered group. For some net $\{f(\alpha), \alpha\epsilon D\}$, $y=o\text{-lim} f$ if and only if $0=o\text{-lim} f_{-y}$, where 0 is an identity of X.*

PROOF. Let $y=o\text{-lim } f$. Then there exist subsets M and N of X such that (i), (ii) and (iii) of it's definition hold. It is easy to see that $M-y=\{m-y| m\epsilon M\}$, $N-y=\{n-y|n\epsilon N\}$ is an up-directed, a down-directed set respectively. It is immediate from (ii) that $0=$l.u.b.$(M-y)=$g.l.b.$(N-y)$. For each $m-y \epsilon M-y$, $n-y\epsilon N-y$, there exists $\beta\epsilon D$ such that $m-y\leqq f(\alpha)-y\leqq n-y$ for all $\alpha\geqq\beta$, i.e., $m-y\leqq f_{-y}(\alpha)\leqq n-y$ for all $\alpha\geqq\beta$. And the converse may be left to the reader.

As an immediate corollary of theorem 2 we have the following.

COROLLARY. *Let X be a partially ordered group. If there exists at least one isolated point of X, then X is a discrete space under it's order topology.*

And we have the following remark.

REMARK. *Let X be a partially ordered group and $\{f(\alpha),\ \alpha\epsilon D\}$ a net in X. If there exists subset M of X such that*

(i) *M is up-directed.*

(ii) *$0=l.u.b.\ M$*

(iii) *for each $m\epsilon M$ there exists $\beta\epsilon D$ such that $m\leqq f(\alpha)\leqq -m$ for all $\alpha\geqq\beta$*

Then $0=o\text{-}lim\ f$.

In fact, it is obvious that $-M=\{-m|m\epsilon M\}$ is a down-directed subset of X. And $0=$g.l.b.$(-M)$. For each $m_1\epsilon M$, $-m_2\epsilon(-M)$ there exists $m_3\epsilon M$ such that $m_3\geqq m_1$, $m_3\geqq m_2$. By the hypothesis (iii) there exists $\beta\epsilon D$ such that $m_1\leqq m_3 \leqq f(\alpha)\leqq -m_3\leqq -m_2$ for all $\alpha\geqq\beta$.

An element a of an l-group is called positive (negative) if $a\geqq 0(a\leqq 0)$. We say that l-group satisfies *chain condition* if every non-void subset of positive elements has a minimal element.

We now prove our main result.

THEOREM 3. *Let X be a complete l-group. X is discrete under it's order topology if and only if X has chain condition.*

PROOF. Suppose X is discrete under it's order topology. Let S be a non-void subset of positive elements. By Zorn's lemma, there exists a maximal chain C in S. Since C is a chain as well as a subset of positive elements, C is lower bounded.

Therefore, by hypothesis there exists $m=\text{g.l.b.}C$. If $m\notin C$, then by lemma 1 $m=o\text{-lim } f$ for some net f, which is contrary to X being discrete. Hence $m\in C \subseteq S$, and moreover m is a minimal element of S. Thus X has chain condition. To prove the converse, suppose X has chain condition. We need only to show that 0 is isolated point, i.e., $X-\{0\}$ is a closed set. Let us assume $X-\{0\}$ is not closed, i.e., there exists a net $\{f(\alpha),\ \alpha\in D\}$ in $X-\{0\}$ such that $0=o\text{-lim } f$. Thus X has two subsets M, N such that (i), (ii) and (iii) are satisfied. It follows that either $0\in N$ implies $0\notin M$, or $0\in M$ implies $0\notin N$. Thus, for any case we have $0\notin N$ or $0\notin M$. Ir first $0\notin N$, since all elemnts of N are positive and N non-void subset of $\{0\}^*-\{0\}$, N has a minimal element s by hypothesis. By (i) N is down directed. Hence the element s is the least element of N. It condradicts to $0=\text{g.l.b. } N$. Next suppose that $0\notin M$. On the other hand, in any l-group the set of positive elements and that of all negative elements are anti-isomorphic. Hence we may see that by chain condition, every non-void subse of negative elements contains a maximal element. Thus by the dual argument M has a greatest element of M. It is also impossible.

By the fact that any complete l-group either satisfies the chain condition, or has at least the cardinal number of the continuum, we have the following.

COROLLARY. *Let X be a complete l-group. If X is not discrete under it's order topology, then X has at least the cardinal number of the continunm.*

From the proof of theorem 3 we have the following.

COROLLARY. *Let X be an l-group. If X has chain condition, then X is discrete under it's order topology.*

November 26, 1961
Mathematical Department
Kyungpook University
Taegu, Korea

REFERENCES

[1] E. S. Wolk, *On order-convergence*, Proc. Amer. Math. Soc. Vol. 12 (1961) pp. 379–384
[2] G. Birkhoff. *Lattice theory*, rev. ed., New York (1948).
[3] __________ *Moore-Smith convergence in general topology*, Ann. of Math. Vol. 38(1937) pp. 39–56
[4] J. L. Kelley, *General topology*, New York, Van Nastrand, 1955.