ON INFINITESIMAL HOLOMORPHICALLY PROJECTIVE TRANSFORMATIONS IN RECURRENT KAEHLERIAN MANIFOLDS

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§ 0. Introduction.

Recently S. Tachibana and S. Ishihara [1], [2] have studied infinitesimal holomorphically projective transformations in Kaehlerian manifolds. They had shown that if a Kaehlerian manifold admits an analytic non-affine infinitesimal holomorphically projective transformation, the Lie derivative of the holomorphically projective curvature tensor vanishes.

T. Sumitomo [3] had shown that the tensor $R_{ki}R_j{}^k+\frac{1}{n}\left(R^{kl}R_{kl}\right)g_{ij}-\frac{2}{n}RR_{ij}$ is positive definite in a non-Einstein Riemannian manifold.

In the present paper we shall investigate infinitesimal holomorphically projective transformations by using above two results in recurrent Kaehlerian manifolds. In §1, we shall give some preliminary facts concerning Kaehlerian manifolds and infinitesimal holomorphically projective transformations for the later use. In §2, we shall show that the Lie derivative of the tensor $P^{kji}{}_h$ not vanishes, and find out a tensor T_{lkji} which is pure for l, i in a recurrent Kaehlerian manifold admitting an analytic non-affine infinitesimal holomorphically projective transformation. In §3, we shall prove that their is no harmonic vector K_l such that $\nabla_l R_{jk}=K_l R_{jk}$ in a compact (Ricci recurrent) Kaehlerian manifold admitting a non-affine infinitesimal holomorphically projective transformation.

§1. Preliminaries.

Let us consider an $n(=2m>2)$ real dimensional Kaehlerian manifold with local coordinates x^i, and we shall restrict our attention in the present paper to manifolds which are real representations of (complex) Kaehlerian manifolds. (Indices run over $1,2,\ldots,n=2m$.)

Then the positive definite Riemannian metric g_{ji} and the complex structure $\varphi_i{}^h$ satisfy the following equations.

$$\varphi_j{}^r\varphi_r{}^i=-\delta_j^i\ , \qquad g_{rt}\varphi_j{}^r\varphi_i{}^t=g_{ji}\ ,$$

$$\varphi_{ji}=g_{ri}\varphi_j{}^r=-\varphi_{ij}\ , \qquad \varphi^{ji}=g^{jr}\varphi_r{}^i=-\varphi^{ij}\ , \tag{1. 1}$$

$$\nabla_k \varphi_j{}^i = 0 \ , \qquad \nabla_k g_{ji} = 0 \ ,$$

where ∇_k denotes the operator of the covariant differentiation with respect to $\left\{ {h \atop j\,i} \right\}$.

Let $R_{kji}{}^h$ be the Riemannian curvature tensor and R_{ji} the Ricci tensor, i.e. $R_{ji} = R_{hji}{}^h$ and $R_{kjih} = R_{kji}{}^r g_{rh}$, $R = R_{ji} g^{ji}$ and

(1. 2) $$S_{ji} = \varphi_j{}^r R_{ri} \ ,$$

then the following identities are valid [4]. (As to the notations, we follow K. Yano [4].)

(1. 3) $$R_{kji}{}^r \varphi_r{}^h = R_{kjr}{}^h \varphi_i{}^r \ , \qquad R_{kjir} \varphi_h{}^r = R_{kjhr} \varphi_i{}^r \ ,$$

(1. 4) $$S_{ji} = -S_{ij} \ .$$

The holomorphically projective curvature tensor $P_{kji}{}^h$ is given by [2]

(1. 5) $$P_{kji}{}^h = R_{kji}{}^h + \frac{1}{n+2}\left(R_{ki} \delta_j{}^h - R_{ji} \delta_k{}^h + S_{ki} \varphi_j{}^h - S_{ji} \varphi_k{}^h + 2 S_{kj} \varphi_i{}^h \right) ,$$

and we can obtain the following identities [1], [4].

(1. 6) $$P_{(kj)i}{}^h = 0 \ , \quad P_{[kji]}{}^h = 0 \ , \quad P_{rji}{}^r = 0 \ , \quad P_{kjr}{}^r = 0 \ ,$$

(1. 7) $$P_{kji}{}^r \varphi_r{}^h = P_{kjr}{}^h \varphi_i{}^r \ , \quad P_{rji}{}^h \varphi_k{}^r = P_{rki}{}^h \varphi_j{}^r \ ,$$

(1. 8) $$P_{rji}{}^t \varphi_t{}^r = 0 \ , \quad P_{kjr}{}^t \varphi_t{}^r = 0 \ .$$

A necessary and sufficient condition for $P_{kji}{}^h = 0$ is that the manifold is a space of constant holomorphic curvature, i.e. a space whose curvature tensor $R_{kji}{}^h$ takes the form

(1. 9) $$R_{kji}{}^h = -\frac{R}{n+2}\left(g_{ki} \delta_j{}^h - g_{ji} \delta_k{}^h + \varphi_{ki} \varphi_j{}^h - \varphi_{ji} \varphi_k{}^h + 2 \varphi_{kj} \varphi_i{}^h \right) .$$

The following identities are known for a vector fields v^i and a tensor field $T_{ji}{}^h$ [4].

(1.10) $$\pounds \nabla_l T_{ji}{}^h - \nabla_l \pounds T_{ji}{}^h = T_{ji}{}^r \pounds \left\{ {h \atop l\,r} \right\} - T_{ri}{}^h \pounds \left\{ {r \atop l\,j} \right\} - T_{jr}{}^h \pounds \left\{ {r \atop l\,i} \right\} \ ,$$

(1.11) $$\nabla_k \pounds \left\{ {h \atop j\,i} \right\} - \nabla_j \pounds \left\{ {h \atop k\,i} \right\} = \pounds R_{kji}{}^h \ ,$$

where $\pounds$ denotes the operator of Lie differentiation with respect to v^i .

We shall call a vector field v^i an infinitesimal holomorphically projective transformation, if it satisfies

(1.12) $$\pounds \left\{ {h \atop j\,i} \right\} = \rho_j \delta_i{}^h + \rho_i \delta_j{}^h - \tilde{\rho}_j \varphi_i{}^h - \tilde{\rho}_i \varphi_j{}^h \ ,$$

where ρ_j is a certain gradient vector and $\tilde{\rho}_j = \varphi_j{}^t \rho_t$.

If we substitute (1.12) into (1.11), then we have

$$(1.13) \qquad \pounds R_{kji}{}^h = \delta_j{}^h \nabla_k \rho_i - \delta_k{}^h \nabla_j \rho_i - \varphi_j{}^h \nabla_k \tilde{\rho}_i + \varphi_k{}^h \nabla_j \tilde{\rho}_i - (\nabla_k \tilde{\rho}_j - \nabla_j \tilde{\rho}_k) \varphi_i{}^h ,$$

and summing for k and h, we find

$$(1.14) \qquad \pounds R_{ji} = -n \nabla_j \rho_i - 2 \varphi_j{}^r \varphi_i{}^t \nabla_r \rho_t . \quad [1] \text{ (pp. 83)}$$

A vector field v^i is called analytic if it satisfies

$$\pounds \varphi_i{}^h \equiv -\varphi_i{}^r \nabla_r v^h + \varphi_r{}^h \nabla_i v^r = 0 .$$

S. Tachibana and S. Ishihara showed that if v^i is an analytic infinitesimal holomorphically projective transformation, the following equation holds [1] (pp. 84)

$$(1.15) \qquad \pounds P_{kji}{}^h = 0 .$$

2. An analytic non-affine infinitesimal holomorphically projective transformation in a recurrent Kaehlerian manifold.

In the present paper we consider an $n(=2m>2)$-dimensional recurrent Kaehlerian manifold defined by [5] (pp. 172)

$$(2.1) \qquad \nabla_l R_{kji}{}^h = K_l R_{kji}{}^h ,$$

where K_l is a vector field.

By the definitions, following relations hold in a recurrent Kaehlerian manifold.

$$(2.2) \qquad \nabla_l R_{ji} = K_l R_{ji} , \qquad \nabla_l P_{kji}{}^h = K_l P_{kji}{}^h ,$$

and from (1.10) and (1.12) we get

$$(2.3) \qquad \begin{aligned} \nabla_k \pounds g_{ji} &= 2\rho_k g_{ji} + \rho_j g_{ki} + \rho_i g_{jk} - \tilde{\rho}_j \varphi_{ki} - \tilde{\rho}_i \varphi_{kj} , \\ \nabla_k \pounds g^{ji} &= -\left(2\rho_k g^{ji} + \rho^j \delta_k{}^i + \rho^i \delta_k{}^j - \tilde{\rho}^j \varphi_k{}^i - \tilde{\rho}^i \varphi_k{}^j\right) . \end{aligned}$$

At first place, we can prove that the following

THEOREM 1. *If a recurrent Kaehlerian manifold admits an analytic non-affine infinitesimal holomorphically projective transformation, the Lie derivative of $P^{kji}{}_h$ not vanishes.*

PROOF. If we assume that $\pounds P^{kji}{}_h = 0$, then by (1.15) we have

$$(2.4) \qquad \begin{aligned} 0 &= \pounds (P_{kji}{}^h P^{kji}{}_h) = \pounds (g_{lh} g^{mi} g^{nj} g^{tk} P_{kji}{}^h P_{tnm}{}^l) \\ &= (\pounds g_{lh}) P_{kji}{}^l P^{kjih} + (\pounds g^{lh}) P_{kjl}{}^i P^{kj}{}_{hi} + 2 (\pounds g^{lh}) P_{lkj}{}^i P_h{}^{kj}{}_i . \end{aligned}$$

Differentiating above equation covariantly, and taking account of (2.2), (2.3) and again making use of (2.4), we can obtain

$$0=(\nabla_r \pounds g_{lh})P_{kji}{}^{l}P^{kjih}+(\nabla_r \pounds g^{lh})P_{kjl}{}^{i}P^{kj}{}_{hi}+2(\nabla_r \pounds g^{lh})P_{lkj}{}^{i}P_{h}{}^{kj}{}_{i}$$

(2. 5)
$$=(2\rho_r g_{lh}+\rho_l g_{rh}+\rho_h g_{rl}-\tilde{\rho}_l\varphi_{rh}-\tilde{\rho}_h\varphi_{rl})P_{kji}{}^{l}P^{kjih}$$
$$-(2\rho_r g^{lh}+\rho^l\delta_r{}^h+\rho^h\delta_r{}^l-\tilde{\rho}^l\varphi_r{}^h-\tilde{\rho}^h\varphi_r{}^l)P_{kjl}{}^{i}P^{kj}{}_{hi}$$
$$-2(2\rho_r g^{lh}+\rho^l\delta_r{}^h+\rho^h\delta_r{}^l-\tilde{\rho}^l\varphi_r{}^h-\tilde{\rho}^h\varphi_r{}^l)P_{lkj}{}^{i}P_{h}{}^{kj}{}_{i}\ .$$

On the other hand, by (1.7), we get

$$\tilde{\rho}_l\varphi_r{}^h P_{kjh}{}^{i}P^{kjl}{}_{i}=\tilde{\rho}_l\varphi_h{}^{i}P_{kjr}{}^{h}P^{kjl}{}_{i}\ ,$$
$$\tilde{\rho}_l\varphi_r{}^h P_{hkj}{}^{i}P^{lkj}{}_{i}=\tilde{\rho}_l\varphi_k{}^h P_{hrj}{}^{i}P^{lkj}{}_{i}\ ,$$

therefore we can obtain from (2.5)

$$4\rho_r P_{lkj}{}^{i}P^{lkj}{}_{i}-2\rho_l P_{kji}{}^{l}P^{kji}{}_{r}+2\rho_l P^{kjli}P_{kjri}+4\rho_l P^{lkji}P_{rkji}$$
$$-4\tilde{\rho}_l\varphi_r{}^h P_{hkj}{}^{i}P^{lkj}{}_{i}-2\tilde{\rho}_l\varphi_r{}^h P_{kjh}{}^{i}P^{kjl}{}_{i}+2\tilde{\rho}_l\varphi_r{}^h P_{kjih}P^{kjil}=0\ ,$$

by some calculations.

Contracting the left hand side of the above equation by ρ^r, we have

(2. 6)
$$4\rho_r\rho^r P_{lkj}{}^{i}P^{lkj}{}_{i}-\rho_l\rho_r(2P_{kji}{}^{l}P^{kji}{}_{r}-2P^{kjli}P_{kjri}-4P^{lkji}P_{rkji})$$
$$-\tilde{\rho}_l\tilde{\rho}^h(4P^{lkj}{}_{i}P_{hkj}{}^{i}+2P^{kjl}{}_{i}P_{kjh}{}^{i}-2P^{kjil}P_{kjih})=0\ .$$

On the other hand, we have the following relations by using (1.7).

a)
$$\tilde{\rho}_l\tilde{\rho}^h P^{kjil}P_{kjih}=\rho_t\varphi_l{}^t P^{kjil}\rho_s\varphi_h{}^s P_{kji}{}^h$$
$$=\rho_t\varphi^{il}P^{kj}{}_{l}{}^{t}\rho_s\varphi_i{}^h P_{kjh}{}^s=\rho_t\rho^s P^{kjht}P_{kjhs}\ ,$$

(2. 7) b)
$$\tilde{\rho}_l\tilde{\rho}^h P^{kjl}{}_{i}P_{kjh}{}^{i}=\rho_t\varphi_l{}^t P^{kjl}{}_{i}(-\rho^s\varphi_s{}^h P_{kjh}{}^{i})$$
$$=\rho_t\varphi_i{}^l P^{kjt}{}_{l}(-\rho^s\varphi^i{}_h P_{kjs}{}^h)=\rho_t\rho^s P^{kjt}{}_{h}P_{kjs}{}^h\ ,$$

c)
$$\tilde{\rho}_l\tilde{\rho}^h P^{lkj}{}_{i}P_{hkj}{}^{i}=\rho_t\varphi_l{}^t P^{lkj}{}_{i}\rho^s\varphi^h{}_s P_{hkj}{}^{i}$$
$$=\rho_t\varphi_l{}^k P^{ltj}{}_{i}\rho^s\varphi^h{}_k P_{hsj}{}^{i}=\rho_t\rho^s P^{htj}{}_{i}P_{hsj}{}^{i}\ .$$

Substituting (2.7) a) b) and c) into (2.6) we obtain

$$\rho_r\rho^r P_{lkj}{}^{i}P^{lkj}{}_{i}=0\ ,$$

from which follows $P_{lkj}{}^{i}=0$ since $\rho_r\neq 0$.

In this case, our manifold is a Kaehler-Einstein space by (1.9), and $R_{ji}=0$ by the first relation of (2.2), and by (1.5) we get $R_{kji}{}^h=0$ which contradicts to our assumption. q.e.d.

Next, Let v^i be an analytic non-affine infinitesimal holomorphically projective transformation. If we substitute (1.12) and (1.15) into the identity

$$\pounds\nabla_l P_{kji}{}^h - \nabla_l \pounds P_{kji}{}^h$$
$$= P_{kji}{}^r \pounds\left\{{h \atop lr}\right\} - P_{rji}{}^h \pounds\left\{{r \atop lk}\right\} - P_{kri}{}^h \pounds\left\{{r \atop lj}\right\} - P_{kjr}{}^h \pounds\left\{{r \atop li}\right\} ,$$

then by using

$$\pounds\nabla_l P_{kji}{}^h = \pounds(K_l P_{kji}{}^h) = (\pounds K_l) P_{kji}{}^h ,$$

we get

$$(\pounds K_l) P_{kji}{}^h = \delta_l^h P_{kji}{}^r \rho_r - 2\rho_l P_{kji}{}^h - \rho_k P_{lji}{}^h - \rho_j P_{kli}{}^h - \rho_i P_{kjl}{}^h$$
$$- \varphi_l{}^h P_{kji}{}^r \tilde{\rho}_r + \varphi_l{}^r (\tilde{\rho}_k P_{rji}{}^h + \tilde{\rho}_j P_{kri}{}^h + \tilde{\rho}_i P_{kjr}{}^h) . \tag{2.8}$$

Contracting on $h=l$, we have

$$(\pounds K_r) P_{kji}{}^r = (n-2)\rho_r P_{kji}{}^r , \tag{2.9}$$

by virtue of (1.6) and (1.8).

By using (1.7) we can see

$$(n-2)\tilde{\rho}_r P_{kji}{}^r = (n-2)\varphi_i{}^r \rho_t P_{kjr}{}^t = \varphi_i{}^r (\pounds K_t) P_{kjr}{}^t$$
$$= \varphi_r{}^t (\pounds K_t) P_{kji}{}^r , \tag{2.10}$$

and by putting

$$\tilde{K}_r = \varphi_r{}^t K_t ,$$

we have

$$(\pounds \tilde{K}_r) P_{kji}{}^r = (n-2)\tilde{\rho}_r P_{kji}{}^r . \tag{2.11}$$

Multiplying the both sides of (2.8) by ρ_h, and substituting (2.9) and (2.11) in it, we have

$$(\pounds K_l + 2\rho_l)(\pounds K_h) P_{kji}{}^h$$
$$= (\pounds K_h)[\rho_l P_{kji}{}^h - \rho_k P_{lji}{}^h - \rho_j P_{kli}{}^h - \rho_i P_{kjl}{}^h \tag{2.12}$$
$$+ \varphi_l{}^r (\tilde{\rho}_k P_{rji}{}^h + \tilde{\rho}_j P_{kri}{}^h + \tilde{\rho}_i P_{kjr}{}^h)] - (\pounds \tilde{K}_h) \tilde{\rho}_l P_{kji}{}^h .$$

On the other hand, from (2.8), we get

$$-2\rho_l P_{kji}{}^h - \rho_k P_{lji}{}^h - \rho_j P_{kli}{}^h - \rho_i P_{kjl}{}^h + \varphi_l{}^r (\tilde{\rho}_k P_{rji}{}^h + \tilde{\rho}_j P_{kri}{}^h$$
$$+ \tilde{\rho}_i P_{kjr}{}^h) = (\pounds K_l) P_{kji}{}^h - \delta_l^h P_{kji}{}^r \rho_r + \varphi_l{}^h \tilde{\rho}_r P_{kji}{}^r . \tag{2.13}$$

Substituting (2.13) into (2.12), we obtain

$$(\pounds K_h)\,\rho_l P_{kji}{}^h-(\pounds K_l)\,\rho_r P_{kji}{}^r+(\pounds \bar{K}_l)\,\bar{\rho}_r P_{kji}{}^r$$
$$-\bar{\rho}_l(\pounds \bar{K}_h)\,P_{kji}{}^h=0\,,$$

and by (2.9), (2.11), above equation is rewriten into

$$[(n-2)\rho_l-\pounds K_l]\rho_r P_{kji}{}^r=[(n-2)\bar{\rho}_l-\pounds \bar{K}_l]\bar{\rho}_r P_{kji}{}^r\,. \tag{2.14}$$

If $\rho_r P_{kji}{}^r=0$, equation (2.8) coincide with equation (4.1) of [1], then it must hold $\rho_l P_{kji}{}^h=0$, which contradicts to our assumption, therefore $\rho_r P_{kji}{}^r\neq 0$. By (1.7) we see

$$\bar{\rho}_r P_{kji}{}^r=\varphi_i{}^r\rho_t P_{kjr}{}^t\,,$$

then from (2.14) we have the following final result.

$$T_{lkji}=\varphi_l{}^t\varphi_i{}^s T_{tkjs}\,, \tag{2.15}$$

where we have put

$$T_{lkji}=[(n-2)\rho_l-\pounds K_l]\rho_r P_{kji}{}^r\,. \tag{2.16}$$

From (2.15) it follows that the tensor T_{lkji} is pure in l, i, [4] hence we have the following

THEOREM 2. *If a recurrent Kaehlerian manifold admits an analytic non-affine infinitesimal holomorphically projective transformation, the tensor* T_{lkji} *defined by* (2,16) *is pure in* l *and* i.

§3. An infinitesimal holomorphically projective transformation in a compact Ricci recurrent Kaehlerian manifold.

In this section we consider a compact Ricci recurrent Kaehlerian manifold defined by

$$\nabla_l R_{ji}=K_l R_{ji}\,, \tag{3. 1}$$

where K_l is a vector field.

Calculating directly we have

$$\nabla_l\nabla_i R_{jk}=\nabla_l(K_i R_{jk})=(\nabla_l K_i)\,R_{jk}+K_i K_l R_{jk}\,,$$
$$\nabla_i\nabla_l R_{jk}=\nabla_i(K_l R_{jk})=(\nabla_i K_l)\,R_{jk}+K_l K_i R_{jk}\,.$$

If we assume that K_l is a harmonic vector field, then we have $\nabla_l K_i=\nabla_i K_l$, and by applying the Ricci's identity to R_{jk}, we obtain

$$R_{lij}{}^r R_{rk}+R_{lik}{}^r R_{jr}=0\,. \tag{3. 2}$$

Let us calculate the Lie derivative of the left side of (3.2) then we have

$$(3.\ 3)\qquad (\pounds R_{lij}{}^{r})R_{rk}+R_{lij}{}^{r}\pounds R_{rk}+(\pounds R_{lik}{}^{r})R_{jr}+R_{lik}{}^{r}\pounds R_{jr}=0\ .$$

If our manifold admits a non-affine (non-analytic) holomorphically projective transformation, by substituting (1.13) and (1.14) into (3.3) and using (1.2), and (1.4), we obtain

$$\begin{aligned}(3.\ 4)\qquad &(\nabla_l\rho_j)R_{ik}-(\nabla_i\rho_j)R_{lk}-(\nabla_l\tilde{\rho}_j)S_{ik}+(\nabla_i\tilde{\rho}_j)S_{lk}\\ &+(\nabla_l\rho_k)R_{ij}-(\nabla_i\rho_k)R_{lj}-(\nabla_l\tilde{\rho}_k)S_{ij}+(\nabla_i\tilde{\rho}_k)S_{lj}\\ &-n(\nabla_r\rho_k)R_{lij}{}^{r}-n(\nabla_r\rho_j)R_{lik}{}^{r}\\ &-2(\nabla_t\rho_s)\varphi_r{}^t\varphi_k{}^sR_{lij}{}^{r}-2(\nabla_t\rho_s)\varphi_r{}^t\varphi_j{}^sR_{lik}{}^{r}=0\ .\end{aligned}$$

From (3.2), we get

$$R_{ilrj}R_k{}^r-R_{lirk}R_j{}^r=0\ ,$$

and contracting it by g^{ji} we have

$$(3.\ 5)\qquad R_{lij}{}^{r}R^{ji}=R_{lt}R^{rt}$$

and using it, we have the following relations by (1.1), (1.2), (1.3) and (1.4).

$$\begin{aligned}&(\nabla_t\rho_s)\varphi_r{}^t\varphi^{ls}R_{lij}{}^{r}R^{ji}=(\nabla_t\rho_s)S^{th}S^s{}_h\\ (3.\ 6)\qquad &=-(\nabla^t\rho_s)\varphi_h{}^k\varphi_l{}^hR_{kt}R^{ls}=(\nabla_t\rho_s)R_k{}^tR^{ks}\ ,\\ &(\nabla_t\rho_s)\varphi_r{}^t\varphi_j{}^sR_{li}{}^{lr}R^{ji}=-(\nabla_t\rho_s)S^t{}_iS^{si}\\ &=(\nabla_t\rho_s)\varphi_i{}^k\varphi^{li}R_{kt}R_l{}^s=-(\nabla_t\rho_s)R_k{}^lR^{ks}\ ,\end{aligned}$$

and

$$\begin{aligned}&\text{a)}\qquad S_l{}^l=0\ ,\\ (3.\ 7)\quad &\text{b)}\qquad (\nabla_r\rho_j)R_{li}{}^{lr}R^{ji}=-(\nabla_r\rho_j)R_t{}^rR^{tj}\ ,\\ &\text{c)}\qquad (\nabla_i\tilde{\rho}_k)S^k{}_jR^{ji}=\varphi_k{}^t(\nabla_i\rho_t)\varphi^{ks}R_{sj}R^{ji}=(\nabla_i\rho_t)R_j{}^tR^{ji}\ .\end{aligned}$$

Contracting (3.4) by $R^{ji}g^{lh}$ and using (3.5), (3.6) and (3.7), we obtain

$$(3.\ 8)\qquad (\nabla_l\rho_j)[R_i{}^lR^{ji}-RR^{lj}]-(\nabla_l\tilde{\rho}_j)S_i{}^lR^{ji}+(\nabla^l\rho_l)R_{ji}R^{ji}=0\ .$$

On the other hand, the following relation hold after the direct calculations.

$$(3.\ 9)\qquad (\nabla_l\tilde{\rho}_j)S_i{}^lR^{ji}=(\nabla_l\rho_t)S_i{}^lS^{ti}=(\nabla_l\rho_t)R_j{}^lR^{tj}\ .$$

Substituting (3.9) into (3.8) we have

$$(3.10)\qquad (\nabla_l\rho_j)RR^{lj}-(\nabla^l\rho_l)R_{ji}R^{ji}=0\ .$$

If we calculate the Lie derivative of the tensor $R_{lij}{}^{r}R^{ji}-R_{lt}R^{rt}$ which

vanishes in a Ricci recurrent Kaehlerian manifold by (3.5) with respect to a holomorphically projective transformation, we have the following equation.

$$(3.11)\qquad (\pounds R_{lij}{}^{r})R^{ji}+R_{lij}{}^{r}\pounds R^{ji}-(\pounds R_{lt})R^{rt}-R_{lt}\pounds R^{rt}=0\ .$$

We can calculate the following relations by making use of (3.5), (1.2), (1.3) and (1.4).

$$(\nabla_h\rho_k)\varphi^{ih}\varphi^{jk}R_{lij}{}^{r}R_r{}^{l}=(\nabla_h\rho_k)S^{hr}S^{k}{}_{r}=(\nabla_h\rho_k)R_t{}^{h}R^{tk}\ ,$$

$$(\nabla_h\rho_k)\varphi_l{}^{h}\varphi_t{}^{k}R^{tr}R_r{}^{l}=(\nabla_h\rho_k)S^{h}{}_{r}S^{kr}=(\nabla_h\rho_k)R_t{}^{h}R^{tk}\ .$$

Substituting (1.13) and (1.14) into (3.11), and multiplying $R_r{}^{l}$ to it, and using (3.5), (3.7) a), (3.9) and above relations, we can obtain the following equation.

$$(3.12)\qquad (\nabla_j\rho_i)RR^{ji}-n(\nabla_l\rho_t)R^{rt}R_r{}^{l}=0\ ,$$

where we have used $S_{ji}R^{ji}=0$.

If we combine (3.10) and (3.12) we can see

$$2(\nabla_l\rho_j)RR^{lj}-n(\nabla_l\rho_t)R^{tr}R_r{}^{l}-(\nabla^l\rho_l)R_{tr}R^{tr}=0\ ,$$

and this equation can be rewriten into

$$(3.13)\qquad (\nabla_l\rho_j)[R_r{}^{l}R^{jr}+\frac{1}{n}g^{lj}(R_{tr}R^{tr})-\frac{2}{n}RR^{lj}]=0\ .$$

By the definition of Lie derivative and (1.12) we have

$$\pounds\{{}^{h}_{ji}\}=\nabla_j\nabla_i v^{h}+R_{rji}{}^{h}=\rho_j\delta_i{}^{h}+\rho_i\delta_j{}^{h}-\tilde{\rho}_j\varphi_i{}^{h}-\tilde{\rho}_i\varphi_j{}^{h}\ ,$$

and summing for i and h, we get

$$(3.14)\qquad \nabla_j(\nabla_i v^i)=(n+2)\rho_j\ ,$$

i.e.

$$(3.15)\qquad \nabla_l\nabla_j(\nabla_i v^i)=(n+2)\nabla_l\rho_j\ .$$

Substituting (3.15) into (3.13) we obtain the following equation.

$$(3.16)\qquad \nabla_l\nabla_j(\nabla_i v^i)[R_r{}^{l}R^{jr}+\frac{1}{n}g^{lj}(R_{tr}R^{tr})-\frac{2}{n}RR^{lj}]=0\ .$$

T. Sumitomo showed that the tensor

$$R_r{}^{l}R^{jr}+\frac{1}{n}g^{lj}(R_{tr}R^{tr})-\frac{2}{n}RR^{lj}$$

is positive definite in a non-Einstein Riemannian manifold [3] (pp. 123), and furtheremore a Ricci (non-symmetric) recurrent Kaehlerian manifold is not an Einstein one.

Therefore, by E. Hopf's theorem [5] we can calculate that the solution function $\nabla_i v^i$ of (3.16) must be constant in a compact Ricci recurrent Kaehlerian manifold.

Again making use of (3.14) we can see that ρ_j vanishes. Then we have the following

THEOREM 3. *In a compact Ricci recurrent Kaehlerian manifold* C_n:

$$\nabla_l R_{ji} = K_l R_{ji} \qquad (K_l \text{ is a vector field.})$$

there exists no harmonic vector field K_l *when* C_n *admits a non-affine (non-analytic) infinitesimal holomorphically projective transformation.*

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