ON ISOMORPHISMS OF LITTLE PROJECTIVE GROUPS OF CAYLEY PLANES

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In the Cayley plane over a Cayley division algebra of characteristic not 2, 3, the little projective gruop Λ is defined to be the group generated by the elations of the plane and is a simple invariant subgroup of the group of all projective transformations of the plane (Jacobson [11]).

The fundamental theorem of projective geometry for Cayley planes is proved by applying the von Staudt's method to a certain four-point. The proof of this theorem is used to classify the involutions in the plane. There are only two kinds of involutions in Λ and any two involutions of the first kind are conjugate within Λ, but not all involutions of the second kind are conjugate. However, there is a canonical form for the involutions of the second kind. The canonical form is used to show that the centralizer of an involution of the first kind in Λ cannot be isomorphic to the centralizer of an involution of the second kind. This is proved by comparing normal series of two subgroups, corresponding to the centralizers, of the norm preserving group $L(J)$, J a reduced exceptional central simple Jordan algebra used to define the plane $\mathfrak{P}$. This result shows that any isomorphism φ of the little projective group Λ_1 of a Cayley plane $\mathfrak{P}_1$ into the little projective group Λ_2 of another plane $\mathfrak{P}_2$ sends an involution of the first kind in Λ_1 to one of the first kind in Λ_2, and hence an elation, which is the product of two involutions of the first kind, to an elation. This leads to the theorem that there exists a projective transformation or correlation $\{\delta\}$ of $\mathfrak{P}_1$ onto $\mathfrak{P}_2$ such that $\{\eta\}^{\varphi}=\{\delta\}^{-1}\{\eta\}\{\delta\}$, $\{\eta\}\in\Lambda_1$. This is the main theorem of this dissertation.

1. Introduction.

R. Moufang in [13] (1933) first discovered a harmonic plane with coordinates

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from any alternative division algebra. Subsequently, it was proved by Bruck and Kleinfield and by Skorniakov independently that the only non-associative alternative division algebra is the Cayley algebra. This implies that the only harmonic non-Desarguesian planes are those based on Cayley algebras. P. Jordan in [12] (1949) and H. Freudenthal in [4] (1951) independently gave definitions of harmonic planes based on an exceptional Jordan algebra over the classical Cayley division algebra (over real numbers). This has been extended recently by T. A. Springer in [15] to construct a Cayley plane by means of any reduced exceptional Jordan algebra $J=\mathcal{H}(\mathcal{C}_3,\gamma)$ where $\mathcal{C}$ is any generalized Cayley division algebra (over a field) of charateristic not 2, 3. Springer proved the fundamental theorem of projective geometry for the planes and established the harmonicity of these planes. In a forthcoming paper [11], N. Jacobson has defined the little projective group of the Cayley plane to be the group generated by the elations and established that this is a simple subgroup. Moreover, he has shown that this group is isomorphic to the factor group of the norm preserving group $L(J)$ by the scalars contained in this group. In the present paper we shall determine the isomorphism of the little projective groups of Cayley planes.

In 2 we give a new proof of the fundamental theorem of projective geometry for Cayley planes by making use of the classical von Staudt's construction. This method is used to classify involution in the little projective group Λ into two classes, i.e. involutions of the first kind and involutions of the second kind in 4 and to study the centralizers of involutions. In 5 we show that an isomorphism of the little projective groups sends any involution of the first kind into an involution of the first kind. This implies that the image of an elation under an isomorphism of one little projective group into a second one is an elation. Finally this result is used to determine the isomorphism of the little projective groups along the line of Schreier and van der Waerden's method of dealing with the analogous result for full linear groups over division rings.

2. The fundamental theorem of projective geometry for Cayley planes.

Let $\mathcal{C}$ be a (generalized) Cayley division algebra over a filed Φ of characteristic not 2, 3. $\mathcal{C}$ has an antiautomorphism of period two $x\to\bar{x}$ over Φ such that $x=\bar{x}$ for x in $\mathcal{C}$ if and only if x is in Φ, and $N(x)=x\bar{x}=\bar{x}x$, $T(x)=x+\bar{x}$. Let $J=\mathcal{H}(\mathcal{C}_3,\gamma)$ be the reduced exceptional simple Jordan algebra consisting of all

3×3 J-Hermitian matrices $X=X^J$ of $\mathscr{C}_3$ with respect to the composition $X\cdot Y=\frac{1}{2}(XY+YX)$, XY the ordinary matrix product in $\mathscr{C}_3$, where $X^J=\gamma^{-1}X^t\gamma$, γ a diagonal matrix with non-zero entries γ_i in $\Phi\cdot X\epsilon J=\mathscr{H}(\mathscr{C}_3,\gamma)$ if and only if

$$(1)\qquad X=\begin{pmatrix} \xi_1 & \gamma_1^{-1}\gamma_2\bar{x} & z \\ x & \xi_2 & \gamma_2^{-1}\gamma_3\bar{y} \\ \gamma_3^{-1}\gamma_1\bar{z} & y & \xi_3 \end{pmatrix},\quad \xi_i\epsilon\Phi,$$

and the generic trace and norm of X are

$$T(X)=\xi_1+\xi_2+\xi_3,$$

$$N(X)=\xi_1\xi_2\xi_3+T((xy)z)-\xi_1\gamma_2^{-1}\gamma_3N(y)$$
$$-\xi_2\gamma_3^{-1}\gamma_1N(z)-\xi_3\gamma_1^{-1}\gamma_2N(x)$$

respectively. Let Π be the set of elements X of rank one in J i.e. $X\times X=O$ where

$$X\times Y=X\cdot Y-\frac{1}{2}T(X)Y-\frac{1}{2}T(Y)X+\frac{1}{2}(T(X)T(Y)-T(X\cdot Y))1.$$

One defines the Cayley (projective) plane $\mathfrak{P}$ as follows: The points and lines of $\mathfrak{P}$ are the rays $\{X\}$, $\{U\}$ of non-zero Φ-multiples of elements X, U of Π. The point $\{X\}$ and the line $\{U\}$ are incident if and only if $(X,U)=0$ for the non-degenerate symmetric bilinear form $(X,Y)\equiv T(X\cdot Y)$ on J. It is known (Springer [15] p. 82) that the Cayley plane $\mathfrak{P}$ is uniquely determined by the Cayley division algebra $\mathscr{C}$ and is independent of the choice of a reduced exceptional simple Jordan algebra $\mathscr{H}(\mathscr{C}_3,\gamma)$.

Let $L(J)$ be the group of 1-1 linear transformations η of J onte J such that $N(X^\eta)=N(X)$, $X\epsilon J$, called the *norm preserving* or *n.p. group* of J.
Jacobson in [11] has shown that the norm preserving group $L(J)$ is generated by the p_{ij}, $p\epsilon\mathscr{C}$, $i\neq j$ where $p_{ij}: X\to P_{ij}XP_{ij}^*$, $p_{ij}=1+pe_{ij}$, e_{ij} the usual matrix unit, $P_{ij}^*=\gamma^{-1}P^{-1}\gamma$. Every element η of $L(J)$ induces a projective transformation $\{\eta\}$ of $\mathfrak{P}$ defined by $\{X\}\to\{X^\eta\}$, $\{X\}$ a point, and $\{U\}\to\{U^{(\eta^*)^{-1}}\}$, $\{U\}$ a line where η^* is the transpose of η relative to (X,Y). An *elation* of $\mathfrak{P}$ is a projective transformation $\neq 1$ leaving every point of a line fixed and every line through a point on the line fixed. The line and point are called the *axis* and *center* of the elation respectively, $\{p_{ij}\}$, $i\neq j$, the projective transformation induced by p_{ij}, is an elation with center $\{e_i\}$ and axis $\{e_j\}$. The *little projective group* Λ is the group generated by elations of $\mathfrak{P}$ and Λ is isomorphic to the factor group $L(J)/\Gamma$ where Γ is the set of scalars $\rho\neq 1$, such that $\rho^3=1$, $\rho\epsilon\Phi$ (Jacobsn[11]).

Let $M(J)$ denote the set of 1-1 linear transformations η in J such that $N(X^\eta)$

$=\rho N(X)$, $\rho \neq 0$ independent of X. Elements of $M(J)$ induce projective transformations in $\mathfrak{P}$ which form a projective group. Following Jacobson we shall call this group the *middle projective group* of $\mathfrak{P}$. By a *four-point* we shall mean a configuration consisting of an ordered quadruple of points no three of which are collinear. Jacobson has proved in [11] that if three points $\{X_i\}$, $i=1,2,3$ are not collinear and three points $\{Y_j\}$, $i=1,2,3$ are not collinear then there is an $\{\eta\}$ in Λ such that $\{X_i\}\{\eta\}=\{Y_i\}$, $i=1,2,3$. Now we consider the case of a four-point:

PROPOSITION 1. *Let* $\{X_i\}$, $i=1,2,3,4$ *and* $\{Y_i\}$, $i=1,2,3,4$ *be two four-points in* $\mathfrak{P}$. *Then there exists a projective transformatiom in the middle projective group which sends the* $\{X_i\}$ *into the* $\{Y_i\}$.

PROOF. Since the Cayley plane $\mathfrak{P}$ is defined independently of the choice of an exceptional Jordan algebra $\mathscr{H}(\mathscr{C}_3, \gamma)$ we assume $\gamma=1$. According to the result quoted above, we may assume that $X_i=e_i=Y_i$, $i=1,2,3$. Also we may take Y_4 to be

$$(2) \qquad A=\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

and we set

$$X_4=\begin{pmatrix} \xi_1 & \bar{x} & z \\ x & \xi_2 & \bar{y} \\ \bar{z} & y & \xi_3 \end{pmatrix}$$

where all the entries of X_4 are not zero since e_i, $i=1,2,3$ and X_4 form a four-point. Let η be a mapping $X\to UXU^*$, $X\epsilon J$, where $U=diag\ \{p,p^{-1},1\}$, $p=z^{-1}$. Since U is unimodular, $\eta\epsilon\ L(J)$ and

$$X_4^{\eta}=\begin{pmatrix} \xi'_1 & \bar{x}' & 1 \\ x' & * & * \\ 1 & * & * \end{pmatrix},$$

$x'\neq 0$. The associated projective transformation $\{\eta\}$ leaves points $\{e_i\}$, $i=1,2,3$ fixed and X_4 is in P_3 for a quadratic subfield P of $\mathscr{C}$ since X_4 is of rank one. Hence we may assume that $X_4\epsilon\mathscr{B}=\mathscr{H}(P_3,\ 1)$, P a quadratic subfield of $\mathscr{C}$. Next we consider a mapping X→UXU*, $X\epsilon\mathscr{B}$, $U=diag\{(\bar{x}')^{-1},\ 1,\ \bar{x}'\}$., Since $N(U)=1$, the mapping is in $L(\mathscr{B})$ and can be extended to an element η in $L(J)$ (Proposition 16, Jacobson [11]). The (1,2)-and (1,3)-entries of $X_4{}^{\eta}$ are 1 so that $X_4\eta\in\mathscr{H}(\Phi_3, 1)$. Since $\{\eta\}$ leaves $\{e_i\}$, $i=1,2,3$ fixed, it suffices to assume that

$$X_4=\begin{pmatrix} 1 & \alpha & \beta \\ \alpha & * & * \\ \beta & * & * \end{pmatrix}$$

where all the entries are non-zero elements of Φ. Let η be a mapping $X\to UXU^*, X\in J$, $U=diag\{1,\alpha^{-1},\beta^{-1}\}$, then $\eta\in M(J)$ since $N(X^\eta)=N(\alpha^{-1}\beta^{-1})N(X)$ as is easily verified. It is easily seen that $X_4{}^\eta=A$ and $\{e_i^\eta\}=\{e_i\}, i=1,2,3$. This proves our assertation.

Let $J_i=\mathscr{H}(\mathscr{C}^{(i)}{}_3, 1), i=1,2$ be two reduced exceptional central simple Jordan algebras over Cayley division algebras $\mathscr{C}^{(i)}, i=1,2$ respectively. Let $e_i, i=1,2,3$ and A be as before in J_1, f_i, $i=1,2,3$ the diagonal idempotents of J_2, and B the element J_2 whose entries are all identity of $\mathscr{C}^{(2)}$. Let $\mathfrak{P}_i$, $i=1,2$ be the Cayley planes of $J_i, i=1,2$, respectively. We shall prove

PROPOSITION 2. *Let $\{\sigma\}$ be a projective transformation of the Cayley plane $\mathfrak{P}_1$ onto the Cayley plane $\mathfrak{P}_2$ such that $\{e_i\}^{\{\sigma\}}=\{f_i\}$, $i=1,2,3$ and $\{A\}^{\{\sigma\}}=\{B\}$. Then there exists a ring-isomorphism θ of $\mathscr{C}^{(1)}$ onto $\mathscr{C}^{(2)}$ such that the mapping $X\to X^\theta$ of J_1 onto J_2 induces $\{\sigma\}$ where X^θ is obtained by applying θ to the entries of X. Moreover $X\to X^\theta$ is a semi-linear transformation of J_1 onto J_2 satisfying $N(X^\theta)=N(X)^s$ where s is the isomorphism of Φ_1 onto Φ_2 associated with $X\to X^\theta$.*

PROOF. Since $\{e_i\}^{\{\sigma\}}=\{f_i\}$, $i=2,3$, $\{\sigma\}$ sends the line $\{e_1\}$ of $\mathfrak{P}_1$ to the line $\{f_1\}$ of $\mathfrak{P}_2$. It is easy to see that the points on the line $\{e_1\}$ are $\{e_3\}$ and points $\{P_p\}$ where

$$P_p=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \bar{p} \\ 0 & p & N(p) \end{pmatrix}, \quad p\in\mathscr{C}^{(1)}. \tag{3}$$

We set $\{P_p\}^{\{\sigma\}}=\{P_{p'}\}$, then since $\{P_p\}^{\{\sigma\}}\neq\{f_3\}$, p' is uniquely determined as an element of $\mathscr{C}^{(2)}$ such that

$$P_{p'}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \bar{p}' \\ 0 & p' & N(p') \end{pmatrix}$$

and hence the mapping $\theta: p\to p'$ is 1-1 onto. We define an addition $\oplus$ and a multiplication $\odot$ among points of the line $\{e_1\}$ by the von Staudt's method as follows:

Take any two points $\{P_p\}$, $\{P_q\}$, different from $\{e_3\}$, on the line $\{e_1\}=\{e_2\}\{e_3\}$ (see Fig. 1(a)) and let the line through $\{P_p\}$ and $\{C\}$, the intersection of two lnies $\{e_1\}\{e_2\}$, $\{e_3\}\{A\}$, meet line $\{e_1\}\{e_3\}$ at the point $\{E\}$.

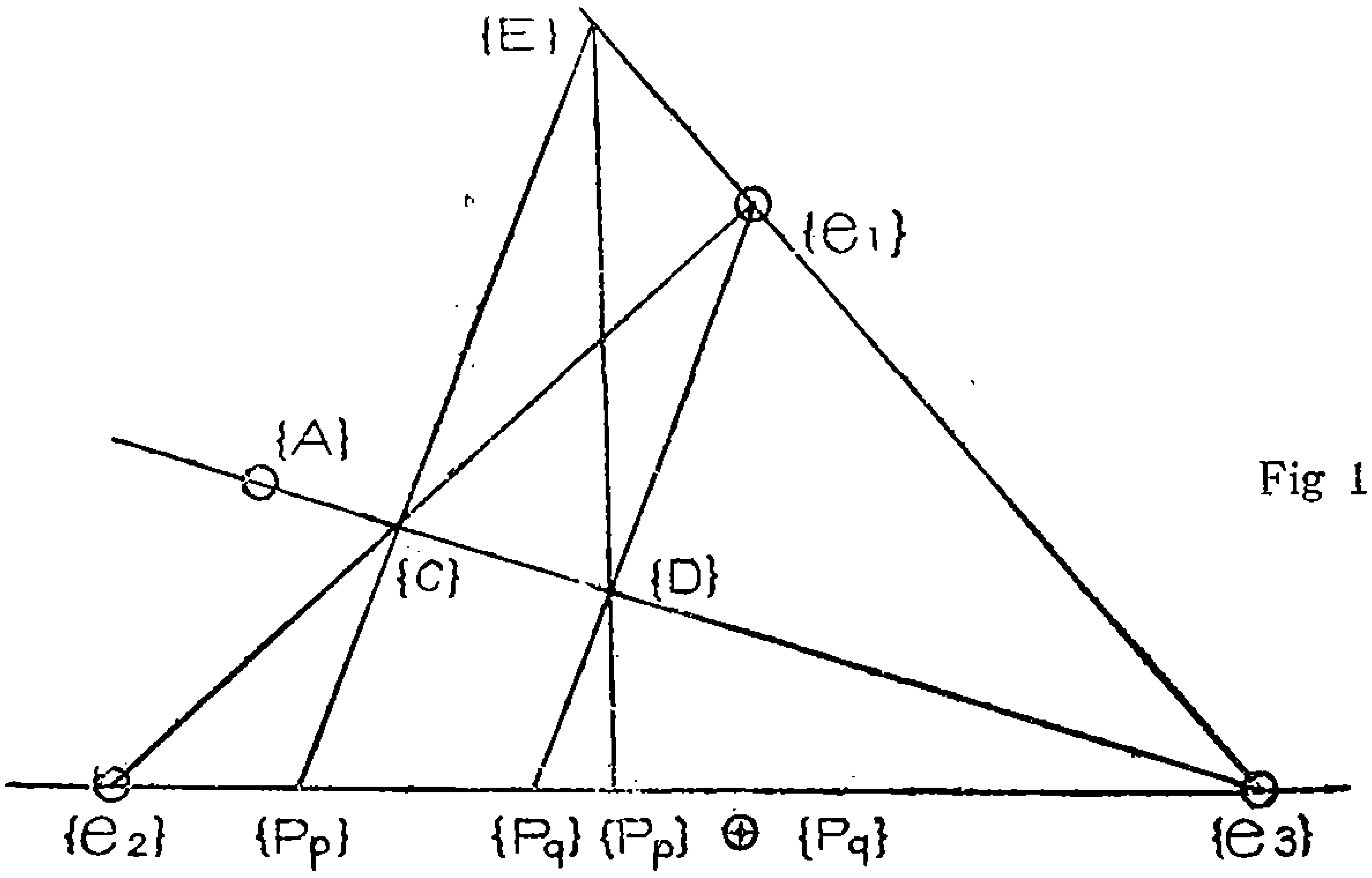

Fig 1 (a)

Let the line $\{P_q\}\{e_1\}$ meet line $\{e_3\}\{A\}$ at the point $\{D\}$, then the line through $\{E\}$ and $\{D\}$ meets line $\{e_1\}$ at a point which is defined to be $\{P_p\}\oplus\{P_q\}$. Similarly $\{P_p\}\odot\{P_q\}$ is defined to be the intersection of lines $\{C\}\{D\}$ and $\{e_1\}$ (see Fig. I(b)).

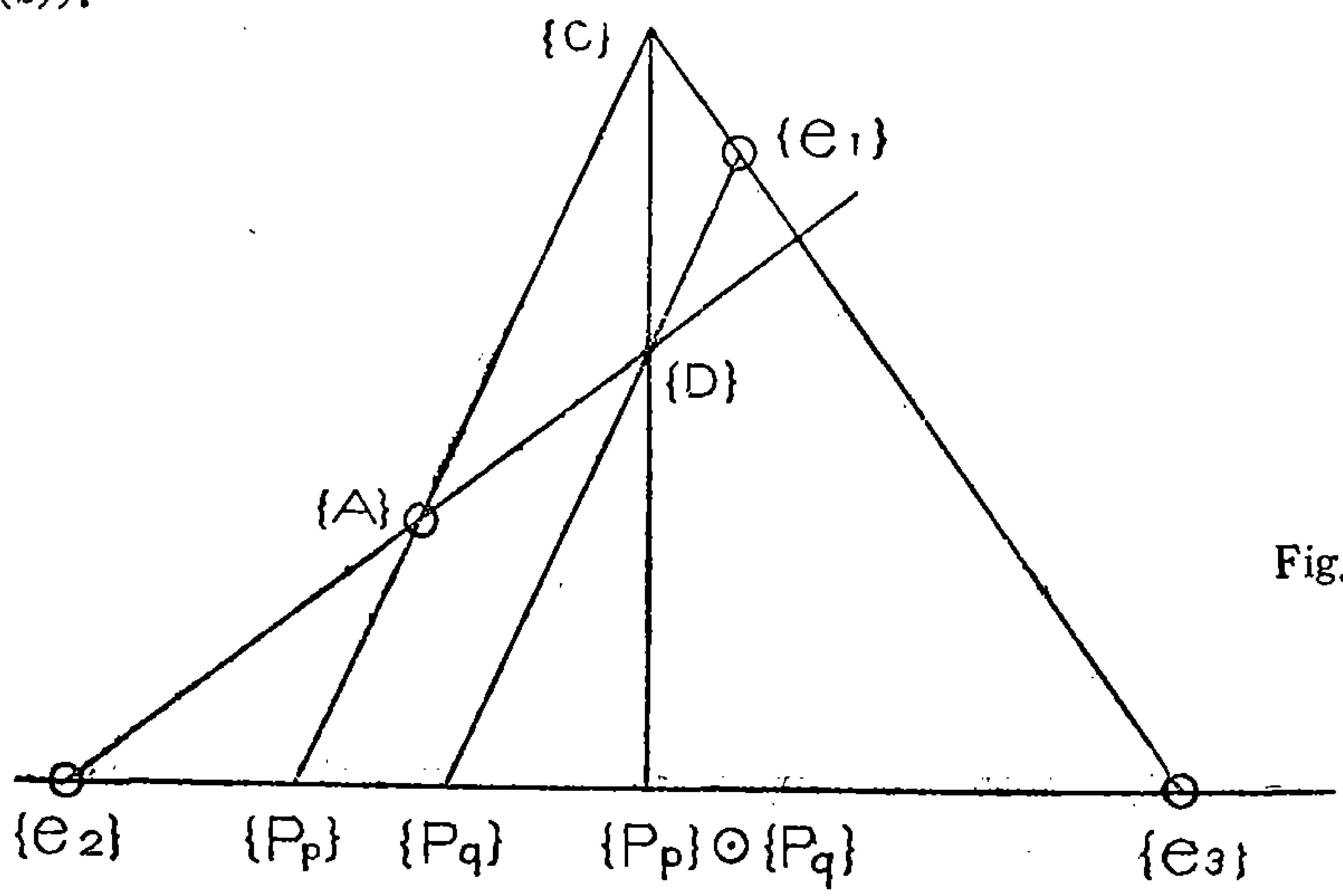

Fig. 1 (b)

We shall show that $\{P_p\}\oplus\{P_q\}=\{P_{p+q}\}$ and $\{P_p\}\odot\{P_q\}=\{P_{pq}\}$ $\{P_p\}$, $\{P_q\}$, points $\neq\{e_3\}$ of the line $\{e_1\}$. We recall (Jacobson [11]) that the line through two points $\{X\}$, $\{Y\}$ is $\{X\times Y\}$ and the intersection of two lines $\{U\}$, $\{V\}$ is $\{U\times V\}$ where $X\times Y=X\cdot Y-\frac{1}{2}T(X)Y-\frac{1}{2}T(Y)X+\frac{1}{2}(T(X)T(Y)-T(X\cdot Y))1$. First we calculate $\{P_p\}+\{P_q\}$. The line $\{e_3\}\{A\}$ in Fig. 1(a) is the ray of non-zero scalar multiples of

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and point $\{C\}$, the intersection of two lines $\{e_3\}=\{e_1\}\{e_2\}$, $\{e_3\}\{A\}$, is the ray of non-zero scalar multiples of

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In the same way we proceed to calculate

$$\{P_p\}\{C\}=\left\{\begin{pmatrix} N(p) & -N(p) & \bar{p} \\ -N(p) & N(p) & -\bar{p} \\ p & -p & 1 \end{pmatrix}\right\}, \qquad \{E\}=\left\{\begin{pmatrix} 1 & 0 & -\bar{p} \\ 0 & 0 & 0 \\ -p & 0 & N(p) \end{pmatrix}\right\}$$

$$\{P_q\}\{e_1\}=\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & N(q) & -\bar{q} \\ 0 & -q & 1 \end{pmatrix}\right\}, \quad \{D\}=\left\{\begin{pmatrix} 1 & 1 & \bar{q} \\ 1 & 1 & \bar{q} \\ q & q & N(q) \end{pmatrix}\right\};$$

$$\{E\}\{D\}=\left\{\begin{pmatrix} * & * & * \\ * & N(p)+N(q)+\frac{1}{2}T(\bar{p}q)+\frac{1}{2}T(p\bar{q}) & -\bar{p}-\bar{q} \\ * & -p-q & 1+\frac{1}{2}T(\bar{p}q)-\frac{1}{2}T(p\bar{q}) \end{pmatrix}\right\},$$

where $*$ in the matrix means an element of $\mathscr{C}^{(1)}$. Finally we arrive at the point $\{P_p\}\oplus\{P_q\}$ which is

$$\left\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \bar{p}+\bar{q} \\ 0 & p+q & N(p+q) \end{pmatrix}\right\}=\{P_{p+q}\},$$

the intersection of two lines $\{e_1\}$, $\{E\}\{D\}$, because of the fact that $T(\bar{p}q)=T(p\bar{q})$ and $N(p+q)=N(p)+N(q)+T(\bar{p}q)$. Next to get $\{P_p\}\odot\{P_q\}$ we follow the same way and we have, in Fig. 1(b),

$$\{P_p\}\{A\}=\{\begin{pmatrix} 1-T(p)+N(p) & p-N(p) & -1+\bar{p} \\ \bar{p}-N(p) & N(p) & -\bar{p} \\ -1+p & -p & 1 \end{pmatrix}\},$$

$$\{C\}=\{\begin{pmatrix} 1 & 0 & 1-\bar{p} \\ 0 & 0 & 0 \\ 1-p & 0 & 1-T(p)+N(p) \end{pmatrix}\};\ \{P_q\}\{e_1\}=\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & N(q) & -\bar{q} \\ 0 & -q & 1 \end{pmatrix}\},$$

$$\{e_2\}\{A\}=\{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix}\},\ \{D\}=\{\begin{pmatrix} N(q) & q & N(q) \\ \bar{q} & 1 & \bar{q} \\ N(q) & q & N(q) \end{pmatrix}\};$$

$$\{C\}\{D\}=\{\begin{pmatrix} * & * & * \\ * & N(pq) & -\bar{q}\bar{p} \\ * & -pq & 1 \end{pmatrix}\}$$

and finally

$$\{P_p\}\odot\{P_q\}=\{\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \overline{pq} \\ 0 & pq & N(pq) \end{pmatrix}\}=\{P_{pq}\}.$$

Since the four-point $\{e_i\}$, $i=1,2,3$ $\{A\}$ are mapping onto the four-point $\{f_i\}$, $i=1,2,3$, $\{B\}$ respectively under $\{\sigma\}$, $(\{P_p\}\oplus\{P_q\})^{(\sigma)}=\{P_p\}^{(\sigma)}\oplus\{P_q\}^{(\sigma)}$, i.e. $\{P_{(pq)'}\}=\{P_{p'}\}\{P_{q'}\}$, which implies $(p+q)'=p'+q'$, and similarly we have $(pq)'=p'q'$. Hence the mapping $\theta: p\to p'$ is a ring-isomorphism of $\mathscr{C}^{(1)}$ onto $\mathscr{C}^{(2)}$, and induces an isomorphism $s: \rho\to\rho'$, of the center $\Phi_1 1$ of $\mathscr{C}^{(1)}$ onto the center $\Phi_2 1$ of $\mathscr{C}^{(2)}$. It follows that $(\rho p)^\theta=\rho^s p^\theta$, $p\in\mathscr{C}^{(1)}$, $\rho\in\Phi_1$. This implies that the mapping $X\to X^\theta$ is a semi-linear transformation of J_1 onto J_2 relative to s and $N(X^\theta)=N(X)^s$. It follows that $X\to X^\theta$ induces a projective transformation $\{X\}\to\{X^\theta\}$ of $\mathfrak{P}_1$ onto $\mathfrak{P}_2$. This projective mapping coincides with $\{\sigma\}$ at the points $\{e_1\}$, $\{A\}$, and every point of the line $\{e_1\}$. It is immediate that two projective transformations which coincide for every point of a line and for two points not on the line are identical. Hence $\{X^\theta\}=\{X\}^{(\sigma)}$, $\{X\}\in\mathfrak{P}_1$.

THEOREM 1. *Let $J_i=\mathscr{H}(\mathscr{C}^{(i)}{}_3,\gamma^{(i)})$, $i=1,2$ be two exceptional Jordan algebras, $\mathfrak{P}_i$, $i=1,2$ the corresponding Cayley planes. Suppose $\{\sigma\}$ is a projcctive iransformation of $\mathfrak{P}_1$ onto $\mathfrak{P}_2$. Then there exists a 1-1 semi-linear transformation σ of J_1 onto J_2 relative to an isomorphism s of Φ_1 onto Φ_2 such that $N(X^\sigma)=N(X)^s$ and σ induces $\{\sigma\}$.*

PROOF. We may assume that $\gamma^{(1)}=1=\gamma^{(2)}$. Take $\{e_i\}$, $i=1,2,3$ and $\{A\}$ in $\mathfrak{P}_1$, and $\{f_i\}$, $i=1,2,3$ and $\{B\}$ in $\mathfrak{P}_2$ as before. Let $\{e_i\}^{(\sigma)}=\{Y_i\}$, $i=1,2,3$ and $\{A\}^{(\sigma)}=\{Y_4\}$, then $\{Y_i\}$, $i=1,2,3,4$ form a four-point in $\mathfrak{P}_2$. By proposition 1 there exists a projective transformation $\{\eta\}$ in the middle projective group of $\mathfrak{P}_2$ such that $\{Y_i\}^{\{\eta\}}=\{f_i\}$, $i=1,2,3$ and $\{Y_4\}^{\{\eta\}}=\{B\}$. Since the mapping $\{\sigma\}\{\eta\}$ sends the four-point $\{e_i\}$, $i=1,2,3$, $\{A\}$ to the four-point $\{f_i\}$, $i=1,2,3$, $\{B\}$, there exists a 1-1 semi-linear transformation σ_1 of J_1 onto J_2 relative to an isomorphism s of Φ_1 onto Φ_2 such that $N(X\sigma_1)=N(X)^s$, $X\epsilon J_1$ and σ_1 induces the mapping $\{\sigma\}\{\eta\}$ (Proposition 2). On the relation $\{\sigma\}=\{\sigma_1\}\{\eta\}^{-1}$, the projective transformation $\{\eta\}^{-1}$ is an element of the middle projective group of $\mathfrak{P}_2$ so that $\{\eta\}^{-1}$ is induced by a 1-1 linear transformation η_1 of J_2 onto J_2 such that $N(Y^{\eta_1})=\rho N(Y)$, $Y\epsilon J_2$, $\rho\neq 0$ in Φ_2. It then follows that $\{\sigma\}$ is induced by a 1-1 semi-linear transformation $\sigma_1\eta_1$ of J_1 onto J_2 relative to an isomorphism s of Φ_1 onto Φ_2 such that $N(X^{\sigma_1\eta_1})=\rho N(X^{\sigma_1})=\rho N(X)^s$, $X\epsilon J_1$. This completes the proof.

Let η be any semi-linear transformation of J relative to an automorphism s of Φ such that $N(X^\eta)=\rho N(X)^s$, $\rho\neq o$ in Φ such that η induces identity on the Cayley plane $\mathfrak{P}$. We shall show that η is a scalar multiple. We may again assume that $\gamma=1$, where $J=\mathscr{H}(\mathscr{C}_3,\gamma)$. We recall (Jacobson [11]) that (X,Y,Z) $=\frac{1}{6}[N(X+Y+Z)-N(X+Y)-N(X+Z)-N(Y+Z)+N(X)+N(Y)+N(Z)]$. It follows that $(X^\eta,Y^\eta,Z^\eta)=\rho(X,Y,Z)^s$, $X,Y,Z\epsilon J$. Set $e_i^\eta=\lambda_i e_i, i=1,2,3$ and $A^\eta=\lambda_4 A$, A as in (2). $(e_1^\eta,e_2^\eta,e_3^\eta)=\rho(e_1,e_2,e_3)^s$ implies that $\lambda_1\lambda_2\lambda_3=\rho$ since s in I on the prime field of Φ, which contains (e_1,e_2,e_3) and (e_i,e_j,A) for all i,j. Similarly we have $\lambda_1\lambda_2\lambda_4=\rho$, $\lambda_1\lambda_3\lambda_4=\rho$ and $\lambda_2\lambda_3\lambda_4=\rho$. These imply $\lambda_1=\lambda_2=\lambda_3=\lambda_4=\lambda$, i.e. $\lambda^3=\rho$. By taking $\lambda^{-1}\eta$ instead of η we may assume $\rho=1$ and $e_i^\eta=e_i$, $i=1,2,3, A^\eta=A$. Then $1^\eta=1$. Theorem 4, Jacobson [9], can be extended to the semi-linear case, i.e. a 1-1 semi-linear transformation σ of J onto J relative to an automorphism t of Φ is a ring-automorphism if and only if $1^\sigma=1$ and $N(X^\sigma)=N(X)^t$, $X\epsilon J$. Hence we see that η is a ring-automorphism. We recall that the decomposition $J=\Phi e_1\oplus\Phi e_2\oplus\Phi e_3\oplus J_{12}\oplus J_{23}\oplus J_{13}$ is the Pierce decomposition of J relative to the e_i where the element a_{ij} of J_{ij} is characterized by $e_j\cdot a_{ij}=\frac{1}{2}a_{ij}=a_{ij}\cdot e_j$, $a_{ij}=ae_{ij}+\bar{a}e_{ij}$ $(\gamma=1)$. Since η is a ring-automorphism of J and $e_i^\eta=e_i$, $i=1,2,3$. we have $J_{ij}^\eta\subseteq J_{ij}$. Hence we can define a semi-linear mapping η_{ij} in $\mathscr{C}$ by $(a^{\eta_{ij}})_{ij}=a_{ij}^\eta, i\neq j$. It follows from $2a_{ij}\cdot b_{jk}=(ab)_{ik}$, i,j,k unequal that $(ab)^{13}=(a^{12})(b^{23})$ By taking $b=1, a=1$ successively we have $\eta_{13}=\eta_{12}, \eta_{13}=\eta_{23}$ because of the fact that $A^\eta=A$ gives $1^{\eta_{ij}}=1, i\neq j$. Set $\theta=\eta_{12}=\eta_{13}=\eta_{23}$, then θ is an an automorphism of $\mathscr{C}$ over the prime field of the base field Φ, and hence η has the form $X\to X^\theta$ where X^θ is obtained by applying θ to the entries of X. Since $\{\eta\}=1$ we have $P_p^\eta=\rho P_p$, $\rho=0$ in Φ and P_p as in (3).

It follows from the of η that $p^{\theta}=p$, $p\epsilon\mathscr{L}$, and so $\theta=1$. Hence $\eta=1$ and we have proved

COROLLARY. *The projective group of the Cayley plane is isomorphic to the factor group of the group of* 1-1 *semi-linear transformations* η *such that* $N(X^{\eta}) =\rho N(X)^{s}$, $X\epsilon J$, $\rho\neq 0$ *in* Φ *and* s *the corresponding automorphism of* Φ, *over the group of the scalar multiplications.*

3. Harmonicity

We shall give here a simple proof of the harmonicity in the Cayley plane $\mathfrak{P}$. A projective transformation of period two in $\mathfrak{P}$ will be called an *involution.* First we introduce a special mapping on J. Let $J=J_0(e_1)\oplus J_1(e_1)\oplus J_{\frac{1}{2}}(e_1)$ be the Peirce decomposition of J relative to the idempotent e_1 where $J_i(e_1)=\{X\epsilon J|e_1\cdot X=iX\}$. We define a mapping ζ on J to be identity on $J_0(e_1)+J_1(e_1)$ and -1 on $J_{\frac{1}{2}}(e_1)$. It is known (Jacobson, [9] p. 185) that the norm preserving group $L(J)$ contains the group $G(J)$ of automorphism of J. ζ is an automorphism of period two in the center of the Galois group $G(J/\Phi e_1)$ of J, the group of automorphism of J leaving e_1 fixed (Jacobson [10] p. 91). Let $\{\zeta\}$ be the projective transformation induced by ζ, an element of $L(J)$, then $\{\zeta\}$ is an involution of the Cayley plane $\mathfrak{P}$ and is an element of the little projective group Λ. We recall that a projective transformation in a projective plane which leaves fixed every point on a line and two points not on the line is the identity. Since ζ leaves fixed every element of $J_0(e_1)+J_1(e_1)$, $\{\zeta\}$ leaves each point of the line $\{e_1\}$ fixed and the point $\{e_1\}$ fixed with no further fixed points.

We note that an elation $\{\tau\}$ has no fixed points outside its axis and $\{\sigma\}^{-1}\{\tau\}\{\sigma\}$, $\{\sigma\}$ a projective transformation of $\mathfrak{P}$, is also an elation. Jacobson in [11] has shown that there exists a projective transformation in Λ mapping a line and a point not on it into another line and a point not on the line respectively.

PROPOSITION 3. *In the Cayley plane there exists a projective transformation which leaves every point of a line fixed, sends a point not on the line into another point not on the line and has no fixed points not on the line.*

PROOF. Using the result just quoted we may assume that the line $\{e_1\}$, the point $\{e_1\}$ and a point $\{X\}$ not on $\{e_1\}$ different from point $\{e_1\}$ are given. It follows (Jacobson [11]) that there exists an elation having $\{e_1\}$ as its axis and sending $\{X\}$ into the point $\{e_1\}$. This proves our assertation.

PROPOSITION 4. *Let $\{U\}$ be a line of $\mathfrak{P}$, and $\{P\}$, $\{Q\}$ two points not on $\{U\}$. There exists a unique involution which leaves every point of $\{U\}$ fixed, sends $\{P\}$ into $\{Q\}$ and has only one fixed point outside $\{U\}$.*

PROOF. Let $\{A\}$, $\{B\}$ be two distinct points which are interchanged by $\{\zeta\}$ defined before. Take two points $\{C\}$, $\{D\}$ on the line $\{e_1\}$ and two points $\{R\}$, $\{S\}$ on the line $\{U\}$ such that $\{A\}$, $\{B\}$, $\{C\}$, $\{D\}$ form a four-point and $\{P\}$, $\{Q\}$, $\{R\}$, $\{S\}$ form a four-point. There is a projective transformation $\{\sigma\}$ mapping $\{A\}$, $\{B\}$, $\{C\}$, $\{D\}$ into $\{P\}$, $\{Q\}$, $\{R\}$, $\{S\}$ respectively. It follows that $\{\eta\} = \{\sigma\}^{-1}\{\zeta\}\{\sigma\}$ is an involution which leaves every point of $\{U\}$ fixed, sends $\{P\}$ to $\{Q\}$ and has only one fixed point $\{e_1\}^{\{\sigma\}}$ not on $\{U\}$. If there is another involution $\{\tau\}$ satisfying the conditions, then $\{\eta\}\{\tau\}^{-1}$ has a line $\{U\}$ of fixed points and leaves fixed two distinct points $\{e_1\}^{\{\sigma\}}$, $\{A\}$ not on $\{U\}$ where we may take $\{B\}$ if $\{e_1\}^{\{\sigma\}} = \{A\}$. This implies that $\{\eta\}\{\tau\}^{-1} = 1$ i.e. $\{\eta\} = \{\tau\}$.

THEOREM 2. *The Cayley plane is harmonic, i.e. the theorem of complete quadrilateral holds.*

PROOF. We follow the proof in Springer [15]. Let $\{P_i\}$, $i=1,2,3$ be three points on a line $\{U\}$ as in Fig. 2. We shall show that there exists a unique point $\{P_4\}$ which is a harmonic conjugate of $\{P_3\}$ relative to $\{P_1\}$ and $\{P_2\}$.

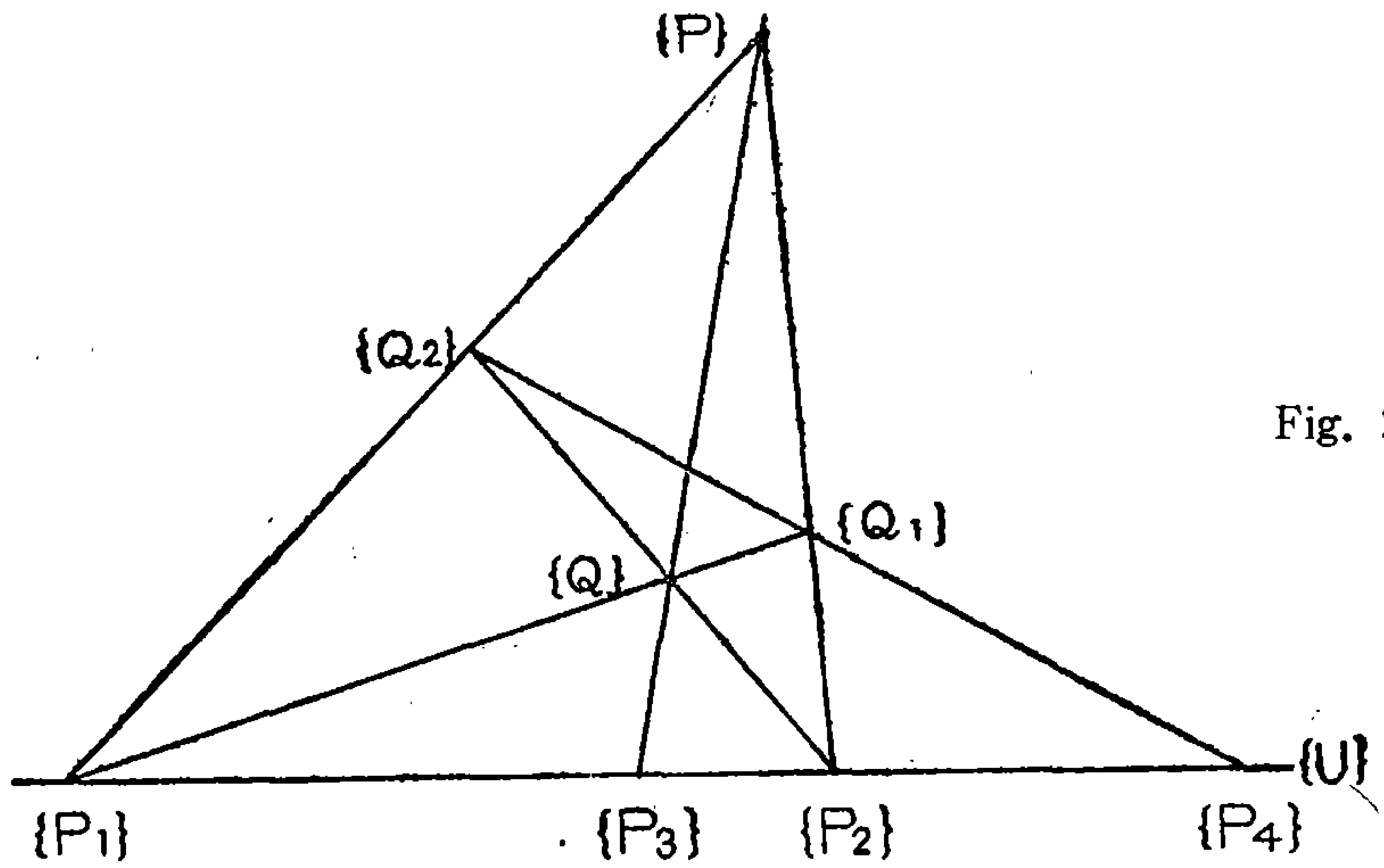

Fig. 2

Let $\{V\}$ be a line through $\{P_3\}$ different from $\{U\}$, and $\{P\}$, $\{Q\}$ two distinct points of $\{V\}$ different from $\{P_3\}$. By proposition 4 there is an involution $\{\eta\}$ which leaves every point of $\{V\}$ fixed and interchanges two points $\{p_1\}$, $\{p_2\}$. Since $\{\eta\}$ sends lines $\{P_1\}\{Q\}$, $\{P_2\}\{P\}$ into lines $\{P_2\}\{Q\}$, $\{P_1\}\{P\}$ respectively, $\{\eta\}$ sends $\{Q_1\}$ to(Q) so that $\{Q_1\}\{Q_2\}$ is a fixed line of $\{\eta\}$. And two fixed lines $\{Q_1\}\{Q_2\}$, $\{P_1\}\{P_2\}$ meet at a fixed point $\{P_4\}$ which is the only one fixed point of $\{\eta\}$ outside of $\{V\}$. It follows that $\{P_4\}$ is independent of the choice of two points $\{P\}$, $\{Q\}$ on $\{V\}$. It remains to show that $\{P_4\}$ is independent of the choice of a line $\{V\}$ through $\{P_3\}$. Let $\{V'\}$ be another line through $\{P_3\}$ and $\{\eta'\}$ the corresponding involution, then there is a projective transformation $\{\tau\}$ which leaves fixed every point of the line $\{U\}$ through the $\{P_i\}$ and sends $\{V\}$ to $\{V'\}$ (Proposition 3). We have $\{\eta\}=\{\tau\}\{\eta'\}\{\tau\}^{-1}$. Since the fixed point $\{P_4'\}$ of $\{\eta'\}$ lies on the line $\{U\}$, we have that $\{\eta\}$ leaves fixed $\{P_4'\}$ different from $\{P_3\}$. Since $\{\eta\}$ has only one fixed point $\{P_4\}$ not on $\{V\}$, $\{P_4'\}$ coincides with $\{P_4\}$. This proves our assertion.

4. Classification of involutions

Let $\{\eta\}$ be any involution different from identity of the little projective group Λ. $\{\eta\}$ has two pairs of interchanging points $\{A\}$, $\{A'\}$; $\{B\}$, $\{B'\}$ which form a four-point. Two invariant lines $\{A\}\{A'\}$, $\{B\}\{B'\}$ meet at a fixed point $\{Z\}$, as in Fig. 3, and lines $\{A\}\{B'\}$, $\{A\}\{B'\}$ meet at fixed points $\{R\}$, $\{S\}$ with corresponding lines $\{A'\}\{B'\}$, $\{A'\}\{B\}$ respectively.

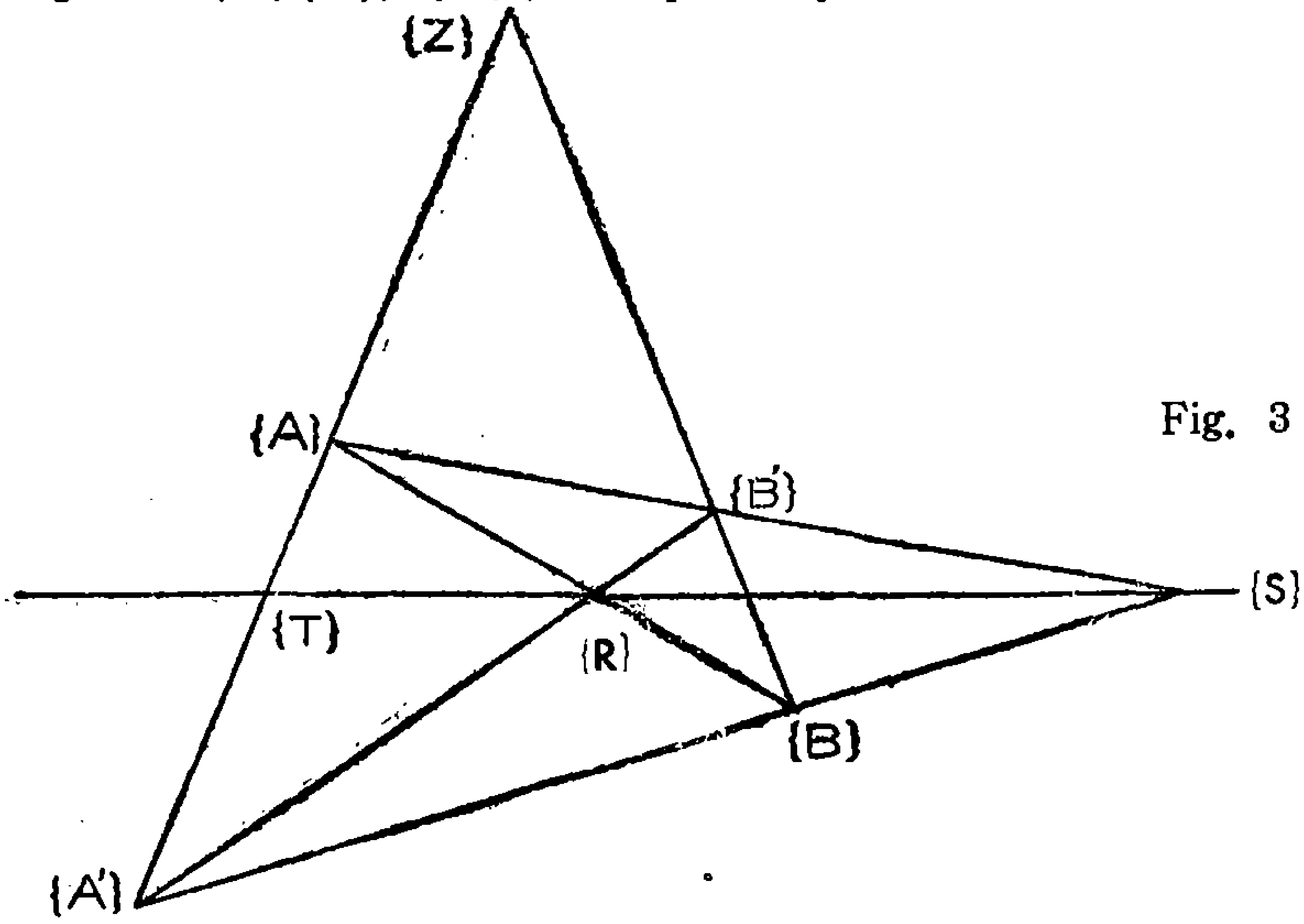

Fig. 3

We have the following two cases:

Case I. $\{\eta\}$ leaves every point of line $\{R\}\{S\}$ fixed,

Case II. $\{\eta\}$ has non-fixed points on line $\{R\}\{S\}$. In case I $\{\eta\}$ has only one fixed $\{Z\}$ outside line $\{R\}\{S\}$ since any projective transformation having a line of fixed pints and two fixed points outside the line is identity. Let invariant lines $\{A\}\{A'\}$, $\{R\}\{S\}$ meet at $\{T\}$, then $\{A'\}$ is the harmonic conjugate of $\{A\}$ relative to two points $\{Z\}$, $\{T\}$. It is easy (Theorem 2) to see that any non-fixed point $\{X\}$ of $\{\eta\}$ is mapped to its harmonic conjugate relative to $\{Z\}$ and the intersection of lines $\{R\}\{S\}$, $\{Z\}\{X\}$. We have seen that $\{\eta\}$ has a line $\{R\}$ $\{S\}$ of fixed points and only one fixed point $\{Z\}$ outside the line, and sends every non-fixed point $\{X\}$ to its harmonic conjugate relative to $\{Z\}$ and the intersection of lines $\{R\}\{S\}$, $\{Z\}\{X\}$. Such an involution will be called *an involution* of *the first kind* (a harmonic homology). The line $\{R\}\{S\}$ and the point $\{Z\}$ are uniquely determined by the involution $\{\eta\}$. These will be called the *axis* and *center* of $\{\eta\}$ respectively. It is immediate that any involution of the first kind is determined completely by its axis and center. An involution of Λ which is not of the first kind, will be called an *involution of the second kind.*

Next we proceed to study an involution $\{\eta\}$ of the second kind, that is, Case II, Let $\{P\}$, $\{P'\}$ be two interchanging points of the line $\{R\}\{S\}$ by $\{\eta\}$ in Fig. 3. Take a line which passes through $\{P\}$, not through $\{Z\}$, the line meets at a fixed point $\{Q\}$ with its image Passing through $\{P'\}$. Since $\{Q\}$ is different from $\{Z\}$, two fixed points $\{Q\}$, $\{Z\}$ form a four-point together with two of three fixed points $\{R\}$, $\{S\}$, $\{T\}$. Hence we have proved the following

PROPOSITION 5. *An involution of the first kind is completely determined by its axis and center. An involution of the second kind has no line of fixed points but a four-point whose elements are fixed.*

We shall characterize an involution $\{\eta\}$ of second kind, and for this purpose we may assume $\gamma=1$ where $J=\mathscr{H}(\mathscr{C}_3,\gamma)$. Let $\{\sigma\}$ be an elment of the middle projective group which sends a four-point left invariant under $\{\eta\}$ onto the four-point $\{e_i\}$, $i=1,2,3$ and $\{A\}$, A as in (2) (Proposition 1). Then the mapping $\{\sigma\}^{-1}\{\eta\}\{\sigma\}$ is an involution of the seconed kind which leaves invariant the four-point $\{e_i\}$, $i=1,2,3$ and $\{A\}$. By Proposition 2 $\{\sigma\}^{-1}\{\eta\}\{\sigma\}$ is induced by a semi-linear transformation $X\rightarrow X^\theta$, $X\epsilon J$ where θ is a ring-automorphism of $\mathscr{C}$ with an associated automorphism of Φ. Since $\{\sigma\}^{-1}\{\eta\}\{\sigma\}$ is in Λ, it is induced by a 1-1 linear transformation τ of J. We have in Corollary, Theorem 1 that the identity of the projective group of $\mathfrak{P}$ is induced by a scalar multiplication $\rho 1, \rho\neq 0$ in Φ. It follows from the fact that the product of τ^{-1} and the mapping $X\rightarrow X^\theta$ induces the identity of the projective group of $\mathfrak{P}$ that the mapping $X\rightarrow X^\theta$ must be linear, that is, θ is an automorphism of $\mathscr{C}$. Furthermore θ is of period two since $\{\sigma\}^{-1}\{\eta\}\{\sigma\}$ is an involution. It is known (Jacobson [8] p.67) that θ decompose $\mathscr{C}$ in the following way; $\mathscr{C}=\vartheta\oplus\vartheta^{\perp}$ where ϑ is a quaternion subalgebra of $\mathscr{C}$, and θ is 1 on ϑ and -1 on $\vartheta^{\perp}$. We denote by ζ_{ϑ} the mapping $X\to X^{\theta}$, $X\epsilon J$. It is easily seen that ζ_{ϑ} is an automorphism of period two of J which belongs to the Galois group $G(J/\mathscr{H}(\vartheta_3,\gamma))$. Therefore we have proved.

PROPOSITION 6. *Any involution of the second kind is conjugate, within the middle projective group, to a projective transformation induced by ζ_{ϑ}, ϑ a quaternion subalgebra of $\mathscr{C}$, where ζ_{ϑ} is an element of $G(J/\mathscr{H}(\vartheta_3,\gamma))$ of period two.*

It should be mentioned here that if one considers involutions of the projective group of all projective transformations of the Cayley plane, there appears one more kind which is neither of the first kind nor of the second kind and which may be called of the third kind. We just note that an involution of the third kind also has a four-point whose elements are fixed.

Now we consider conjugacy among involutions of the first kind in the following

PROPOSITION 7. *Any two involution of the first kind are conjugate to each other within the little projective group Λ.*

PROOF. Let $\{\eta_i\}$, $i=1,2$ be two given involutions of the first kind with axes $\{U_i\}$, $i=1,2$ and centers $\{Z_i\}$, $i=1,2$ respectively. It follows (Jacobson [11]) that there exists an element $\{\sigma\}$ in Λ such that $\{U_1\}^{\{\sigma\}}=\{U_2\}$ and $\{Z_1\}^{\{\sigma\}}=\{Z_2\}$. This implies $\{\eta_2\}=\{\sigma\}^{-1}\{\eta_1\}\{\sigma\}$ and the result is proved.

We note that involution $\{\zeta\}$ defined in 3 is of the first kind with axis $\{e_1\}$ and center $\{e_1\}$ and is conjugate to any involution of the first kind within Λ.

5. Isomorphic images of involutions

We have defined the automorphism ζ of period two of J to be 1 on $J_0(e_1)+J_1(e_1)$ and -1 on $J_{\frac{1}{2}}(e_1)$ where $J_i(e_1)=\{X\epsilon J\,|\,e_1\cdot X=iX\}$, $i=0,1,1/2$, and have seen that the projective transformation $\{\zeta\}$ induced by ζ is an involution of the first kind in the little projective group Λ which is conjugate to any involution of the first kind. Let ζ_{ϑ} be the mapping $X\to X^{\theta}$, $X\epsilon J$ given in proposition 6 where θ is an automorphism of period two of $\mathscr{C}$ and has the decomposition $\mathscr{C}=\vartheta\oplus\vartheta^{\perp}$, ϑ a quaternion subalgebra of $\mathscr{C}$ such that θ is 1 on ϑ and -1 on $\vartheta^{\perp}$, We have seen that ζ_{ϑ} is an automorphism of period two of J belonging to $G(J/\mathscr{H}(\vartheta_3,\gamma))$ and induces an involution $\{\zeta_{\vartheta}\}$ of the second kind. Let $\mathscr{L}(\{\zeta\})$, $\mathscr{L}(\{\zeta_{\vartheta}\})$ be centralizers of involutions $\{\zeta\}$, $\{\zeta_{\vartheta}\}$ in the little projective group Λ. Our first step is to show that these two centralizers are not isomorphic. We note that ζ and ζ_{ϑ} are linear transformations of period two in 27-dimensional vector space $J=\mathscr{H}(\mathscr{C}_3,\gamma)$. It follows that ζ and ζ_{ϑ} have decompositions $J=J^+\oplus J^-$ and $J=J^+_{\vartheta}\oplus J^-_{\vartheta}$ respectively such that ζ and ζ_{ϑ} are I on J^+ and J^+_{ϑ}, and $-I$ on J^- and J^-_{ϑ}, where $J^+=J_0(e_1)+J_1(e_1)$, $J^-=J_{\frac{1}{2}}(e_1)$, $J^+_{\vartheta}=\mathscr{H}(\vartheta_3,\gamma)$ and J^-_{ϑ} the set of

$$(4)\qquad \begin{pmatrix} 0 & \gamma_1^{-1}\gamma_2 x & z \\ x & 0 & \gamma_2^{-1}\gamma_3 y \\ \gamma^{-1}\gamma_3\bar{z} & y & 0 \end{pmatrix},\quad x, y, z\in\vartheta^{\perp}.$$

It is clear that a linear transformation commutes with an involutive linear transformation η if and only if it leaves invariant the plus- and minus-subspaces of η. Let $C(\eta)$, $C(\zeta_{\vartheta})$ be centralizers of ζ, ζ_{ϑ} in $L(J)$ respectively. It follows that $C(\zeta)$ is the set $\{\eta\,|\,\eta\in L(J),\ (J^+)^{\eta}\subseteq J^+$ *and* $(J^-)^{\eta}\subseteq J^-\}$ and $C(\zeta_{\vartheta})$ is $\{\eta\,|\,\eta\in L(J),\ (J^+_{\vartheta})^{\eta}\subseteq J^+_{\vartheta}$ and $(J^-_{\vartheta})^{\eta}\subseteq J^-_{\vartheta}\}$. They contain Γ, the set of scalars $\rho 1$ such that $\rho^3=1$, $\rho\in\Phi$. We shall prove

PROPOSITION 8. *Let $\mathscr{L}(\{\zeta\})$, $\mathscr{L}(\{\zeta_{\vartheta}\})$ be the centralizers of $\{\zeta\}$, $\{\zeta_{\vartheta}\}$ respectively in Λ. Then $\mathscr{L}(\{\zeta\})=C(\zeta)/\Gamma$ and $\mathscr{L}(\{\zeta_{\vartheta}\})=C(\zeta_{\vartheta})/\Gamma$.*

PROOF. It is clear that every element of $C(\zeta)$ induces an element of $\mathscr{L}(\{\zeta\})$. So we prove that the mapping $\eta\to\{\eta\}$ is a homomorphism of $C(\zeta)$ onto $\mathscr{L}(\{\zeta\})$. Let $\{\eta\}$ be any element of $\mathscr{L}(\{\zeta\})$, i.e. $\{\eta\}\{\zeta\}=\{\zeta\}\{\eta\}$, and η an element of $L(J)$ which induces $\{\eta\}$. Then $\{X^{\eta\zeta}\}=\{X^{\zeta\eta}\}$, $X\in J$ and so $\eta\zeta=\omega\zeta\eta$, $\omega\neq 0$ in Φ, $\omega^3=1$. Since $\zeta^2=1$ and $\eta\zeta^2=\omega\zeta\eta\zeta=\omega^2\zeta^2\eta$, i.e. $\eta=\omega^2\eta$, we have $\omega^2=1$ and hence $\omega=1$. Hence $\eta\in C(\zeta)$, and $\mathscr{L}(\{\zeta\})=C(\zeta)/\Gamma$ since the kernel of the mapping of $C(\zeta)$ onto $\mathscr{L}(\{\zeta\})$ is Γ. By the same argument we have $\mathscr{L}(\{\zeta_{\vartheta}\})\cong C(\zeta_{\vartheta})/\Gamma$.

PROPOSITION 9. *Let $C(\zeta)$, $C(\zeta_{\vartheta})$ be the centralizers of ζ, ζ_{ϑ} in $L(J)$ respectively where ζ, ζ_{ϑ} are as before. Then $C(\zeta)$ and $C(\zeta_{\vartheta})$ are not isomorphic to each other.*

The proof will be made by comparing normal series of the groups $C(\zeta)$, $C(\zeta_{\vartheta})$. A projective transformation $\{\zeta\}$ is in $\mathscr{L}(\{\zeta\})$ if and only if $\{\eta\}$ leaves the point $\{e_1\}$ and the line $\{e_1\}$ fixed because of the fact that $\{\eta\}^{-1}\{\zeta\}\{\eta\}=\{\zeta\}$ and $\{\zeta\}$ leaves fixed the point $\{e_1\}$ and each point of the line $\{e_1\}$. It follows from

Proposition 8 and the above remark that any element ζ of $C(\zeta)$ satisfies $e_1\eta = \rho e_1$, $\rho \neq 0$ in Φ and $J_0(e_1)\eta \subseteq J_0(e_1)$. Set $H = \{\eta \in L(J) \mid e_1\eta = e_1$ and $J_0\eta \subseteq J_0\}$, then H is an invariant subgroup of $C(\zeta)$ containing the commutator subgroup of $C(\zeta)$ so that the factor group $C(\zeta)/H$ is abelian. We recall that the Peirce component $J_0(e_1)$ relative to a primitive idempotent e_1 is the Jordan algebra $\Phi(e_2+e_3)+\mathfrak{M}$. $\mathfrak{M} = \Phi(e_2-e_3)+J_{23}$ of the non-degenerate bilinear form in $\mathfrak{M}$ given by $(X,Y)_0 = \frac{1}{2}(X,Y)$, $X, Y \in \mathfrak{M}$ where (X,Y) is the bilinear form of J, and J_0 has the generic norm $N_0(A) = \xi^2 - (X,X)_0$, $A = \xi + X$, $\xi \in \Phi$, $X \in \mathfrak{M}$ (Jacobson [10] p. 84). Set $\theta(A) = e_1 + A$, $A \in J_0$, then $N(\theta(A)) = N_0(A)$ and $L_{\theta(A)}$, $A \in J_0$, maps the Peirce component $J_{\frac{1}{2}}$ into itself where U_A is defined as $2R_A^2 - R_{A^2}$, R_A the mapping $X \to X \cdot A$. Let K be the set of

$$\eta = U_{\theta(B_1)} U_{\theta(B_2)} \cdots\cdots\cdots U_{\theta(B_r)}$$

where $\prod_{i=1}^{r} N(\theta(B_i)) = 1$, $B_i \in J_0$. Then K is a subgroup of the reduced n. p. group $L_2(J)$ which is contained in the n. p. group $L(J)$, and moreover K is a subgroup of H because of the fact that $U_{\theta(A)}$, $A \in J_0$ leaves fixed e_1 and J_0. We shall show that K is an invariant subgroup of H and the factor group H/K is abelian. To do this we define a mapping $\lambda: \eta \to \eta_0$, $\eta \in H$ where η_0 is the restriction of η to the Peirce component $J_0(e_1)$. Since $N_0(A^{\eta_0}) = N(\theta(A^{\eta_0})) = N(\theta(A)^{\eta}) = N(\theta(A)) = N_0(A)$, $A \in J_0$, $\eta \in H$, we have $\eta_0 \in O(J_0, N_0)$, the orthogonal group of J_0 relative to N_0, for any $\eta \in H$ so that the mapping λ is a homomorphism of H into $O(J_0, N_0)$. It is known (Jacobson [9] p. 187) that if J_0 is the Jordan algebra of the bilinear form $(X,Y)_0$ as before then the reduced orthogonal group $O'(J_0, N_0)$ is the group $L_2(J_0)$ of elements of form $U_{B_1}U_{B_2}\cdots\cdots U_{B_r}$, $\prod_{1}^{r} N_0(B_i) = 1$. It then follows from this and definition of K that the mapping λ is a homomorphism of K onto $O'(J_0, N_0)$. The argument in the proof of Theorem 6, [10] shows that any η of $L(J)$ which is 1 on $J_0 + J_1$ is either 1 or ζ defined in 3. Hence the kernel of the mapping λ consists of 1 and ζ. It is known (Jacobson [10] pp. 89-91) that the Galois group $G(J/\Phi e_1)$ is the set of $U_{\theta(x_1)}U_{\theta(x_2)}\cdots U_{\theta(x_{2r})}$ such that $\prod_{1}^{2r} N(\theta(X_i)) = 1$ and $X_i \in \mathfrak{M}$. It follows that the group K contains $G(J/\Phi e_1)$ and hence ζ where ζ is in the center of $G(J/e_1)$. So we have seen that $H/\{1, \zeta\} \cong H^{\lambda}$ the subgroup of $O(J_0, N_0)$ and $K/\{1, \zeta\} \cong O'(J_0, N_0)$. Here we recall that J_0 has a positive Witt index. It follows that $O'(J_0, N_0)$ is the commutator subgroup of $O(J_0, N_0)$ (Chevalley 2, p. 53). Since $O'(J_0, N_0)$ is an invariant subgroup of H^{λ} and K is a subgroup of H we can conclude that K is an invariant subgroup of H. Hence the factor group H/K which is isomorphic to $H^{\lambda}/O'(J_0, N_0)$ is abelian. It is known (Dieudonne [3] p. 58) that the factor group of $O'(J_0, N_0)$ over its center is simple. Since the center of $O'(J_0, N_0)$ is abelian, the normal series of $C(\xi): C(\zeta), H, K, 1$ has only one non-solvable factor group which is simple.

Next we consider a normal series of $C(\zeta_{\vartheta})$. We define a mapping $\lambda: \eta \rightarrow \eta_0$, $\eta \in C(\zeta_{\vartheta})$ where η, is the restriction of η to the plus-subspace $\mathscr{H}(\vartheta_3, \gamma)$ of ζ_{ϑ}. λ is welldefined because of the fact any η of $C(\zeta_{\vartheta})$ leaves invariant the subpsace $\mathscr{H}(\vartheta_3, \gamma)$ of ζ_{ϑ}. And λ is a homomorphism of $C(\zeta_{\vartheta})$ into the n.p. group $L(\mathscr{H}(\vartheta_3, \gamma))$ of $\mathscr{H}(\vartheta_3, \gamma)$. The kernel of λ is the Galois group $G(J/\mathscr{H}(\vartheta_3, \gamma))$ over $\mathscr{H}(\vartheta_3, \gamma))$; for, any element of the kernel of λ leaves fixed the identity of $J = \mathscr{H}(\mathscr{C}_3, \gamma)$ so that it is an autimorphism of J which belohgs to $G(J/\mathscr{H}(\vartheta_3, \gamma))$. Since any element of $G(J/\mathscr{H}(\vartheta_3, \gamma))$ leaves invariant both plus- and minus-subspaces of ζ_{ϑ}, it is contained in $C(\zeta_{\vartheta})$, and hence the kernel of λ coincides with $G(J/\mathscr{H}(\vartheta_3, \gamma))$. We have a normal series of the centralizer $C(\zeta_{\vartheta})$:

$$C(\zeta_{\vartheta}),\ G(J/\mathscr{H}(\vartheta_3, \gamma)),\ 1.$$

We shall show that factor groups of this series are not solvable. We recall that on the decomposition $\mathscr{C} = \vartheta \oplus \vartheta^{\perp}$ by the automorphism θ of period two of $\mathscr{C}$, we have $\vartheta\vartheta^{\perp} \subseteq \vartheta^{\perp}$ and $\vartheta^{\perp}\vartheta \subseteq \vartheta^{\perp}$, and the trace of any element of $\vartheta^{\perp}$ is zero. Let S be the group generated by the p_{ij}, $i \neq j$, $p \in \vartheta$ where $p_{ij}: X \rightarrow P_{ij} X P^*_{ij}$, $p_{ij} = 1 + p e_{ij}$. We shall show that S is contained in $C(\zeta_{\vartheta})$. Take, for instance, p_{12}, $p \in \vartheta$, then for any X of the minus-subspace J_{ϑ} of ζ_{ϑ} we have

$$X^{p_{12}} = \begin{pmatrix} 0 & \gamma_1^{-1}\gamma_2\bar{x} & z \\ x & 0 & \gamma_2^{-1}\gamma_3\bar{y} \\ \gamma_3^{-1}\gamma_1\bar{z} & y & 0 \end{pmatrix}^{p_{12}}$$

$$= \begin{pmatrix} px + \bar{x}\bar{p} & \gamma_1^{-1}\gamma_2\bar{x} & z + \gamma_2^{-1}\gamma_2 p\bar{y} \\ x & 0 & \gamma_2^{-1}\gamma_3\bar{y} \\ \gamma_3^{-1}\gamma_1\bar{z} + \gamma_2^{-1}\gamma_1 y\bar{p}\ y & & 0 \end{pmatrix}.$$

Since $px \in \vartheta^{\perp}$ and the (1,1)-entry of $X^{p_{12}}$ is zero, we have $X^{p_{12}} \in J^{-}_{\vartheta}$, $X \in J^{-}_{\vartheta}$ so that p_{12} leaves invariant $\mathscr{H}(\vartheta_3, \gamma)$ and J_{ϑ}. Hence $p_{12} \in C(\zeta_{\vartheta})$, $P \in \vartheta$. It follows by the same argument that the p_{ij}, $i \neq j$, $p \in \vartheta$ are contained in $C(\zeta_{\vartheta})$, i.e. $S \subseteq C(\zeta_{\vartheta})$. It follows that the image S^{λ} of S under the homomorphism λ is contained in $C(\zeta_{\vartheta})^{\lambda}$. We note that the p_{ij}, $i \neq j$ generate the unimodular group $SL_3(\vartheta)$. It follows that S^{λ} is anti-isomorphic to $SL_3(\vartheta)$ by the mapping $p_{ij}^{\lambda} \rightarrow P_{ij}$. That is, $C(\zeta_{\vartheta})^{\lambda}$ contains a subgroup S^{λ} which is anti-isomorphic to the unimodular group $SL_3(\vartheta)$. It is known (Dieudonne [3] p.38) that the projective unimodular group $PSL_3(\vartheta)$, the factor group of $SL_3(\vartheta)$ by its center, is simple. Hence S^{λ} is not solvable. It follows that $C(\zeta_{\vartheta})^{\lambda}$ is not solvable. Therefore the factor group $C(\zeta_{\vartheta})/G(J/\mathscr{H}(\vartheta_3, \gamma))$ which is isomorphism to $C(\zeta_{\vartheta})^{\lambda}$ is not solvable.

It is known (Jacobson [10] p. 88) that $G(J/\mathscr{H}(\mathfrak{g}_3,\gamma))$ is isomorphic to the multiplicative group U of elements of norm 1 of $\mathfrak{g}$ which contains the commutator subgroup of the multiplicative group of $\mathfrak{g}$. Since the multiplicative group of a division ring is not solvable (Hua [7] p. 1), the group U is not solvable. Hence we have proved that the normal series of $C(\zeta_{\mathfrak{g}})$ has at least two non-solvable factor groups. On the other hand we have seen that the normal series of $C(\zeta)$ has only one non-solvable simple factor group. It follows from Schreier's refinement theorem that two groups $C(\zeta)$, $C(\zeta_{\mathfrak{g}})$ cannot be isomorphic to each other. This proves Proposition 9.

Now we take up the centralizers $\mathscr{C}(\{\zeta\})$ and $\mathscr{C}(\{\zeta_{\mathfrak{g}}\})$. In the proof of Proposition 9 we have seen that $C(\zeta)$ has a normal series with only one non-solvable simple factor group while $C(\zeta_{\mathfrak{g}})$ has a normal series with at least two non-solvable factor groups. Hence normal series $C(\zeta)$, Γ, 1 and $C(\zeta_{\mathfrak{g}})$, Γ, 1 have refinements having the corresponding properties, that is, the normal series $C(\zeta)$, Γ, 1 has only one non-solvable simple factor group between $C(\zeta)$ and Γ while the normal series $C(\zeta_{\mathfrak{g}})$, Γ, 1 has at least two non-solvable factor groups between $C(\zeta_{\mathfrak{g}})$ and Γ. A comparison of normal series of two groups $C(\zeta)/\Gamma$ and $C(\zeta_{\mathfrak{g}})/\Gamma$ gives us the following

PROPOSITION 10. *Let $\mathscr{C}(\{\zeta\})$, $\mathscr{C}(\{\zeta_{\mathfrak{g}}\})$ be the centralizers of $\{\zeta\}$, $\{\zeta_{\mathfrak{g}}\}$ in Λ respectively. Then $\mathscr{C}(\{\zeta\})$ cannot be isomorphic to $\mathscr{C}(\{\zeta_{\mathfrak{g}}\})$.*

Let $J_i=\mathscr{H}(\mathscr{C}_3^{(i)},\gamma^{(i)})$, $i=1,2$ be exceptional simple Jordan algebras. $\mathfrak{P}_i$, $i=1,2$ the corresponding Cayley planes, and Λ_i, $i=1,2$ the corresponding little projective groups. Let φ be any isomorphism of Λ_1 onto Λ_2. We shall show that φ maps any involution of the first kind of Λ_1 into an involution of the first kind of Λ_2 It follows from Proposition 10 that the image of the involution $\{\eta\}$ of Λ_1 defined in 3 cannot be an involution $\{\zeta_{\mathfrak{g}}\}$ of the second kind of Λ_2 defined in 4. We have seen that any involution of the first kind of Λ_1 is conjugate to $\{\zeta\}$ (Proposition 7) and any involution of the second kind of Λ_2 is conjugate to $\{\zeta_{\mathfrak{g}}\}$ for some quarternion subalgebra $\mathfrak{g}$ of $\mathscr{C}^{(2)}$ (Proposition 6). It follows then that the isomorphic image of any involution of the first kind of Λ_1 cannot be an involution of the second kind of Λ_2. Hence we have proved

THEOREM 3. *Any isomorphism φ of Λ_1 onto Λ_2 maps an involution of the first kind of Λ_1 into an involution of the first kind of Λ_2.*

6. Isomorphic images of elations

From now on we are only interested in involution of the first kind, so we shall call them *involutions* simply. We shall introduce the following

PROPOSITION 11. (Coxeter [1]). *Any two involutions in the Cayley plane are commutative if and only if the center of each lies on the axis of the other.*

PROOF. Let $\{\eta_i\}$, $i=1,2$ be two involutions in $\mathfrak{P}$, and $\{Z_i\}$, $\{U_i\}$, $i=1,2$ centers and axes of the $\{\eta_i\}$ respectively. First we shall prove the sufficiency. Thus suppose that $\{\eta_1\}$ and $\{\eta_2\}$ are involutions such that the center of each lies on the axis of the other. It is easily seen that $\{\eta_1\}\{\eta_2\}$ leaves fixed each point of the line $\{Z_1\}\{Z_2\}$ and each line through the intersection $\{P\}$ of $\{U_1\}$ and $\{U_2\}$. Take a point $\{X\}$ on $\{U_1\}$ different from $\{Z_2\}$, $\{P\}$, then $\{\eta_1\}\{\eta_2\}$ sends $\{X\}$ into the harmonic conjugate of $\{X\}$ relative to $\{P\}$ and $\{Z_2\}$. We have seen that the mapping $\{\eta_1\}\{\eta_2\}$ of Λ has center $\{P\}$ and axis $\{Z_1\}\{Z_2\}$ and sends a non-fixed point $\{X\}$ into the harmonic conjugate of $\{X\}$ relative to $\{P\}$ and the intersection of the axis and $\{P\}\{X\}$. Hence $\{\eta_1\}\{\eta_2\}$ is also an involution. Similarly we know that $\{\eta_2\}\{\eta_1\}$ is an involution with center $\{P\}$ and axis $\{Z_1\}\{Z_2\}$. It follows from the fact that these involutions coincide on two lines $\{U_1\}$, $\{U_2\}$ that $\{\eta_1\}\{\eta_2\}=\{\eta_2\}\{\eta_1\}$. Conversely, suppose that $\{\eta_1\}$ and $\{\eta_2\}$ are commutative, then the product $\{\eta_1\}\{\eta_2\}$ is a projective transformation of Λ of period two. $\{\eta_2\}$ leaves invariant $\{U_1\}$ since $\{X\}^{\{\eta_2\}\{\eta_1\}}=\{X\}^{\{\eta_1\}\{\eta_2\}}=\{X\}^{\{\eta_2\}}$, $\{X\}\in\{U_1\}$. Hence the axis $\{U_1\}$ either passes through the center $\{Z_2\}$ or coincides with the axis $\{U_2\}$. We note that the product of two involutions with the same axis is an elation and an elation is not a projective transformation of period two. It follows that $\{U_1\}$ cannot coincide with $\{U_2\}$. Hence $\{U_1\}$ passes through $\{Z_2\}$ and similarly $\{U_2\}$ through $\{Z_1\}$.

Let $\mathfrak{I}$ be a set of involution in Λ and we denote by $c(\mathfrak{I})$ the set of involutions of Λ which commutes with all elements of $\mathfrak{I}$.

PROPOSITION 12. *Let* $\{\eta_1\}$, $\{\eta_2\}$ *be any two non-commutative involutions (of the first kind) in the Cayley plane* $\mathfrak{P}$. *A necessary and sufficient condition for* $\{\eta_1\}$, $\{\eta_2\}$ *to have either centers or axes in common is that* $c(c(\{\sigma_1\},\{\sigma_2\}))$ $=c(c(\{\eta_1\},\{\eta_2\}))$ *for each pair of non-commuative involutions* $\{\sigma_1\}$, $\{\sigma_2\}$ *of* $c(c(\{\eta_1\}\ \{\eta_2\}))$.

PROOF. Let $\{Z_i\}$, $\{U_i\}$, $i=1,2$ be centers and axes of $\{\eta_i\}$, $i=1,2$ respectively. Suppose first that $\{Z_1\}=\{Z_2\}=\{Z\}$, then, by Proposition 11, $c(\{\eta_1\},\{\eta_2\})$ consists of involutions whose common center is the intersection $\{P\}$ of $\{U_1\}$ and $\{U_2\}$ and whose axes are lines through $\{Z\}$. It follows that $c(c(\{\eta_1\},\{\eta_2\}))$ consists of involutions having $\{Z\}$ as common center and lines through $\{P\}$ as axes. Hence non-commutative $\{\sigma_1\}$, $\{\sigma_2\}$ of $c(c(\{\eta_1\}, \{\eta_2\}))$ have a common center $\{Z\}$ and axes through $\{P\}$. It is easily seen that $c(c(\{\sigma_1\}, \{\sigma_2\}))$ coincides with $c(c(\{\eta_1\}, \{\eta_2\}))$. Dually we have the necessity for the case of common axis. In order to prove the sufficiency we suppose that $\{Z_1\} \neq \{Z_2\}$ and $\{U_1\} \neq \{U_2\}$. Since $c(\{\eta_1\}, \{\eta_2\})$ consists of only one involution having the intersection $\{P\}$ of $\{U_1\}$ and $\{U_2\}$ as center and line $\{Z_1\}\{Z_2\}$ as axis, $c(c(\{\eta_1\}, \{\eta_2\}))$ consists of involutions whose centers are points on the line $\{Z_1\}\{Z_2\}$ and whose axes are lines through $\{P\}$. If we take two elements $\{\sigma_1\}$, $\{\sigma_2\}$ of $c(c(\{\eta_1\}, \{\eta_2\}))$ such that $\{\sigma_1\}$ and $\{\sigma_2\}$ have their centers in common, then $c(c(\{\sigma_1\}, \{\sigma_2\}))$ is contained properly in $c(c(\{\eta_1\}, \{\eta_2\}))$ since the line $\{Z_1\}\{Z_2\}$ contains at least three points. Our result is proved.

Now we are ready to consider isomorphic images of elations under φ where φ is an isomorphism of the little projective group Λ_1 onto the little projective group Λ_2. In any Cayley plane an elation $\{\tau\}$ with axis $\{U\}$ may be expressed as the product of two involutions $\{\eta_1\}$, $\{\eta_2\}$ of the little projective group having same axis $\{U\}$ (Coxeter [1] p. 63). Indeed, let $\{X\}$, $\{X'\}$ be two points not on $\{U\}$ such that $\{X\}^{\{\tau\}} = \{X'\}$ and $\{P\}$ the intersection of $\{U\}$ and line $\{X\}\{X'\}$. We set $\{Q\}$ to be the harmonic conjugate of $\{P\}$ relative to points $\{X\}$, $\{X'\}$ then involutions $\{\eta_1\}$, $\{\eta_2\}$ with centers $\{X\}$, $\{Q\}$ respectively will have the desired property.

THEOREM 4. *Let φ be any isomorphism of Λ_1 onto Λ_2. φ maps any elation of Λ_1 into an elation of Λ_2.*

PROOF. Let $\{\tau\}$ be an elation of Λ_1, then there are two involutions $\{\eta_1\}$, $\{\eta_2\}$ of Λ_1 such that $\{\tau\} = \{\eta_1\}\{\eta_2\}$, and $\{\eta_1\}$, $\{\eta_2\}$ have the same axis in common as that of $\{\tau\}$. It follows from Proposition 12 that $\{\eta_1\}^{\varphi}$ and $\{\eta_2\}^{\varphi}$ have either their centers or their axes in common. It is easily seen that the product of two iovolutions having either axes or centers in common is an elation. Hence $\{\tau\}^{\varphi} = \{\eta_1\}^{\varphi}\{\eta_2\}^{\varphi}$ is an elation.

7. Determination of an isomorphism

PROPOSITION 13. (*Hall* [6]). *If a projective transformation $\{\sigma\} \neq 1$ has a line $\{U\}$ of fixed points, then there exists a point $\{Z\}$ such that $\{\sigma\}$ leaves $\{Z\}$ and eaery line through $\{Z\}$ fixed and has no further fixed points or lines. Dually, if a projective transformation $\{\sigma\} \neq 1$ leaves fixed a point $\{Z\}$ and each line through it, then there exists a line $\{U\}$ such that $\{\sigma\}$ leaves fixed each* *point of $\{U\}$ and has no further fixed points or lines.*

PROOF. It follows from the fact that a projective transformation leaving invariant each point of a line and two points not on the line is the identity that there exists at most one fixed point of $\{\sigma\}$ outside the line $\{U\}$. First let $\{\sigma\}$ have a fixed point $\{Z\}$ not on $\{U\}$, then every line through $\{Z\}$ meets $\{U\}$ at a fixed point so that it is fixed under $\{\sigma\}$. Let $\{V\}$ be a fixed line besides $\{U\}$ and all lines through $\{Z\}$, then each point of $\{V\}$ is a fixed point as the intersection of $\{V\}$ and a fixed line through $\{Z\}$, and hence $\{\sigma\}$ has two lines of fixed points, which contradicts to $\{\sigma\} \neq 1$. Next suppose that $\{\sigma\}$ has no fixed point not on $\{U\}$. Let $\{P\}$ be any point not on $\{U\}$ and $\{Z\}$ the intersection of $\{U\}$ and line $\{P\}\{P\}^{\{\sigma\}}$. Then the line $\{P\}\{P\}^{\{\sigma\}}$ is invariant under $\{\sigma\}$, for, the line $\{P\}\{Z\}$ which is the same line as $\{P\}\{P\}^{\{\sigma\}}$ is mapped to the line through points $\{P\}^{\{\sigma\}}$ and $\{Z\}=\{Z\}^{\{\sigma\}}$. Since points $\{P\}$, $\{P\}^{\{\sigma\}}$ and $\{Z\}$ are collinear, the line $\{P\}^{\{\sigma\}}\{Z\}$, the image of line $\{P\}\{Z\}$, coincides with the line $\{P\}\{Z\}$ which is $\{P\}\{P\}^{\{\sigma\}}$. Since there is no fixed point not on $\{U\}$, $\{P\}$ lies on no fixed lines besides $\{P\}\{P\}^{\{\sigma\}}$. Let $\{V\}$ be any fixed line $\neq \{U\}$, then $\{V\}$ must meet the fixed line $\{P\}\{P\}^{\{\sigma\}}$ at the fixed point $\{Z\}$ because of the fact that there is no fixed point not on $\{U\}$ under $\{\sigma\}$. Take any line $\{W\} \neq \{U\}$ through $\{Z\}$ and a point $\{Q\}$ on $\{W\}$ different from $\{Z\}$. Since $\{Q\}$ not on $\{U\}$ lies on a unique fixed line $\{Q\}\{Q\}^{\{\sigma\}}$ which passes through the point $\{Z\}$, the fixed line coincides with $\{W\}$. Hence $\{\sigma\}$ leaves fixed $\{Z\}$ and every line through $\{Z\}$. The rest of our assertions follow by duality.

We shall use the method of Schreier and van der Waerden [14] in determining an isomorphism φ. First we consider subgroups of the little projective group in the Cayley plane $\mathfrak{P}$ which consist of identity and elations. Jacobson has proved in [11] that an elation is determined by its axis, any point not on the axis, and the image of the point.

PROPOSITION 14. *In the Cayley plane $\mathfrak{P}$ let $\{\tau_i\}$, $i=1,2$ be elations with centers $\{C_i\}$ and axes $\{U_i\}$, $i=1,2$ respectively. A necessary and sufficient condition that the product $\{\tau\}$ of $\{\tau_1\}$ and $\{\tau_2\}$ be an elation or identity is either $\{C_1\}=\{C_2\}$ or $\{U_1\}=\{U_2\}$.*

PROOF. First let us prove the sufficiency. If $\{C_1\}=\{C_2\}=\{C\}$ and $\{\tau\}\neq 1$, then $\{\tau\}$ leaves the point $\{C\}$ and all lines through it fixed so that there exists a line $\{V\}$ of fixed points of $\{\tau\}$ by Proposition 13. We shall show that the product $\{\tau\}$ of two elations $\{\tau_1\}$, $\{\tau_2\}$ has no fixed point $\neq \{C\}$ on the lines $\{U_i\}$,

$i=1,2$. Let $\{X\}$ be any point $\neq \{C\}$ on $\{U_1\}$, then $\{X\}^{\{\tau\}}=\{X\}^{\{\tau_1\}\{\tau_2\}}=\{X\}^{\{\tau_2\}}$ since $\{U_1\}$ is the axis of the elation $\{\tau_1\}$. It follows from the fact that an elation $\{\tau_2\}$ has no fixed points outside its axis $\{U_2\}$ that $\{X\}^{\{\tau_2\}}\neq\{X\}$ i.e. $\{X\}^{\{\tau\}}\neq\{X\}$. Likewise any point $\neq\{C\}$ of $\{U_2\}$ cannot be fixed under $\{\tau\}$. It follows that the line $\{V\}$ of fixed points of $\{\tau\}$ cannot meet the lines $\{U_1\}$, $\{U_2\}$ at points different from $\{C\}$, i.e. the line $\{V\}$ passes through $\{C\}$. Hence the product $\{\tau\}$ of $\{\tau_1\}$ and $\{\tau_2\}$ is an elation with center $\{C\}$ and axis $\{V\}$. If $\{U_1\}=\{U_2\}=\{U_3\}=\{U\}$ and $\{\tau\}\neq 1$, then $\{U\}$ is the line of fixed points of $\{\tau\}$. Furthermore $\{\tau\}$ has no fixed points putside $\{U\}$. Indeed, if there is a fixed point $\{X\}$ not on $\{U\}$ of $\{\tau\}=\{\tau_1\}\{\tau_2\}$ then the point $\{X\}$ is mapped to $\{X\}^{\{\tau_1\}}$ under $\{\tau_1\}$, and then $\{X\}^{\{\tau_1\}}$ is mapped back to $\{X\}\ \{X\}^{\{\tau_1\}\{\tau_2\}}$. This implies that the elation $\{\tau_2\}$ is the inverse of $\{\tau_1\}$ because of the fact that any elation is determined by its axis, a point not on the axis and the image of the point (here $\{U\}$, $\{X\}^{\{\tau_1\}}$, $\{X\}$ respectively). We then have that $\{\tau_1\}\{\tau_2\}$ is the identity, i.e. $\{\tau\}=1$ which contradicts our assumption $\{\tau\}\neq 1$. Since $\{\tau\}$ has a line $\{U\}$ of fixed points, by Proposition 13 $\{\tau\}$ has a fixed point $\{C\}$ such that every line through $\{C\}$ is fixed. This point $\{C\}$ lies on $\{U\}$ since $\{\tau\}$ has no fixed point outside $\{U\}$. Hence $\{\tau\}$ is an elation with center $\{C\}$ and axis $\{U\}$. In order to prove the necessity we suppose that the product $\{\tau\}$ of two elations $\{\tau_1\}$, $\{\tau_2\}$ is not identity and the condition fails, i.e. $\{C_1\}\neq\{C_2\}$ and $\{U_1\}\neq\{U_2\}$. It suffices to show that $\{\tau\}$ is not an elation. The intersection $\{P\}$ of $\{U_1\}$ and $\{U_2\}$ is a fixed point of $\{\tau\}$ which is different from $\{C_1\}$, $\{C_2\}$ and so is not on the line $\{C_1\}\{C_2\}$. We have seen at the beginning of the proof that the product $\{\tau\}=\{\tau_1\}\{\tau_2\}$ has no fixed point on the lines $\{U_1\}$, $\{U_2\}$ except the intersection $\{P\}$. Let $\{V\}$ be any line through $\{P\}$ being different from the $\{U_i\}$, $i=1,2$ and $\{X\}$ a point $\neq\{P\}$ of $\{V\}$ not on the line $\{C_1\}\{C_2\}$. We may take such a point $\{X\}$ of $\{V\}$ because $\{P\}$ is not on the line $\{C_1\}\{C_2\}$. The line $\{C_1\}\{X\}$ is different from the line through $\{C_2\}$ and $\{X\}^{\{\tau_1\}}$, the image of $\{X\}$ under the elation $\{\tau_1\}$. Hence the image $\{X\}^{\{\tau\}}$ of $\{X\}^{\{\tau_1\}}$ under the elation $\{\tau_2\}$ cannot be on the line $\{C_1\}\{X\}$ so that $\{X\}^{\{\tau\}}$ cannot be $\{X\}$. So we have seen that not all points of the line $\{V\}$ can be fixed under $\{\tau\}$. Therefore we know that $\{\tau\}$ has no line of fixed points through the fixed point $\{P\}$. Since any fixed point of an elation lies on its axis, the product $\{\tau\}$ is not an elation. We note that if the product $\{\tau\}$ of two elations $\{\tau_1\}$, $\{\tau_2\}$ is the identity then the elations $\{\tau_1\}$, $\{\tau_2\}$ have the same center and axis.

Now let $\Lambda_{\{X\}}$ be a subgroup of the little projective group Λ which consists of identity and elations. It follows from Proposition 14 that all elements of $\Lambda\{X\}$ must have in common either center $\{P\}$ or axis $\{U\}$. So we have two kinds of subgroups, that is, a subgroup $\Lambda\{P\}$ which consists of 1 and elations having center $\{P\}$ in common and a subgroup $\Lambda\{U\}$ which consists of 1 and elations having axis $\{U\}$ in common. We note that there are 1-1 correspondences between points of $\mathfrak{P}$ and subgroups $\Lambda\{P\}$, $\{P\}$ a point, and between lines of $\mathfrak{P}$ and subgroups $\Lambda\{U\}$, $\{U\}$ a line. Given any two subgroups $\Lambda\{P\}$, $\Lambda\{Q\}$, $\{P\}$ and $\{Q\}$ points, there exists $\{\sigma\}$ in the little projective group Λ such that $\{P\}^{\{\sigma\}} = \{Q\}$ (Jacobson [11]) so that $\Lambda\{Q\} = \{\sigma\}^{-1}\Lambda\{P\}\{\sigma\}$. It follows that subgroups $\Lambda\{P\}$, $\{P\}$ a point, form a class of cojugate subgroups in Λ, and similarly subgroups $\Lambda\{U\}$, $\{U\}$ a line, form a class of conjugate subgroups in Λ. It follows from the fact that a projective transformation does not map a point to a line, or a line to a point that $\Lambda\{P\}$, $\{P\}$ a point, is not conjugate to $\Lambda\{U\}$, $\{U\}$ a line, in the projective group of all projective transformations of $\mathfrak{P}$.

Now we turn to the general case, let $\mathfrak{P}_i$, $i=1,2$ be two Cayley planes, and φ an isomorphism of the little projective group Λ_1 of $\mathfrak{P}_1$ onto the little projective group Λ_2 of $\mathfrak{P}_2$. Let $\Lambda\{X\}$ be a subgroup of Λ_1 defined before. The set $\{\{\tau\}^{\varphi} | \{\tau\} \in \Lambda\{X\}\}$ is then a subgroup of Λ_2 which consists of 1 and elations of Λ_2 by Theorem 4. It follows from Proposition 14 that the set is of the form $\Lambda\{X\}'$ where $\{X\}'$ is a point or a line of $\mathfrak{P}_2$. We denote this set by $\Lambda_{\{X\}}^{\bar{\varphi}}$, then $\bar{\varphi}$ is a mapping of the set of subgroups $\Lambda\{X\}$, $\{X\} \in \mathfrak{P}_1$ onto the set of subgroups $\Lambda\{X\}'$, $\{X\}' \in \mathfrak{P}_2$ which is induced by φ. We have seen that in the little projective group there are two classes of conjugate subgroups $\Lambda\{X\}$, $\{X\}$ an element of the Cayley plane, that is, a conjugate class of subgroups $\Lambda\{P\}$, $\{P\}$ a point and a conjugate class of subgroups $\Lambda\{U\}$, $\{U\}$ a line. It follows that the mapping $\bar{\varphi}$ either preserves the types of conjugate classes, or interchanges them. Set $\Lambda_{\{X\}}^{\bar{\varphi}} = \Lambda\{X\}'$, $\{X\} \in \mathfrak{P}_1$, then the mapping $\{X\} \to \{X\}'$ in either 1-1 of the set of points (lines) of $\mathfrak{P}_1$ onto the set of points (lines) of $\mathfrak{P}_2$, or 1-1 of the set of points (lines) of $\mathfrak{P}_1$ onto the set of lines (points) of $\mathfrak{P}_2$. We shall show that two subgroups $\Lambda\{P\}$, $\{P\}$ a point and $\Lambda\{U\}$, $\{U\}$ a line have an elation in common if and only if $\{P\} \in \{U\}$. If an elation $\{\tau\}$ is in both subgroups then $\{\tau\}$ has its center $\{P\}$ and axis $\{U\}$ so that $\{P\} \in \{U\}$. Conversely, if $\{P\} \in \{U\}$ then an elation with center $\{P\}$ and axis $\{U\}$ which is conjugate to the elation $\{p_{12}\}$ with center $\{e_1\}$ and axis $\{e_2\}$ is a common element of the subgroups $\Lambda\{P\}$, $\Lambda\{U\}$. Hence the mapping $\{X\} \to \{X\}'$ preserves the incidence relation so that it is a projective transformation or correlation of $\mathfrak{P}_1$ onto $\mathfrak{P}_2$. We shall denote this mapping by $\{\delta\}$.

Given any element $\{\eta\}$ of Λ_1 we have $\{\eta\}^{-1}(\Lambda_{\{X\}})\{\eta\}=\Lambda_{\{X\}^{\{\eta\}}}$, $\{X\}\in\mathfrak{P}_1$. Taking the image of this under φ we get $(\{\eta\}^{\varphi})^{-1}(\Lambda_{\{X\}^{\{\delta\}}})\{\eta\}\varphi=\Lambda_{\{X\}^{\{\eta\}\{\delta\}}}$. Since the left hand side of this is $\Lambda_{\{X\}^{\{\delta\}\{\eta\}^{\varphi}}}$, we have $\{X\}^{\{\eta\}\{\delta\}}=\{X\}^{\{\delta\}\{\eta\}^{\varphi}}$, $\{X\}\in\mathfrak{P}_1$ so that $\{\eta\}^{\varphi}=\{\delta\}^{-1}\{\eta\}\{\delta\}$, $\{\eta\}\in\Lambda_1$. We have proved the main

THEOREM 5. *Let Λ_i, $i=1,2$ be the little projective groups of Cayley planes $\mathfrak{P}_i$, $i=1,2$ respectively and φ any isomorphism of Λ_1 onto Λ_2. There exists a projective transformation or correlation $\{\delta\}$ of $\mathfrak{P}_1$ onto $\mathfrak{P}_2$ such that* $\{\eta\}^{\varphi}=\{\delta\}^{-1}\{\eta\}\{\delta\}$, $\{\eta\}\in\Lambda_1$.

As a special case of Theorem 5 we have the following

COROLLARY. Let φ be any automorphism of the little projective group Λ of a Cayley plane $\mathfrak{P}$. Then there exists a projective transformation or correlation $\{\delta\}$ of $\mathfrak{P}$ such that $\{\eta\}=\{\delta\}^{-1}\{\eta\}\{\delta\}$, $\{\eta\}\in\Lambda$.

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