CONFORMAL COLLINEATIONS IN RECURRENT SPACES.

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§1. Conformal Collineation.

Let M be a Riemannian space with metric tensor g_{jk} and the Christoffel symbols Γ^i_{jk}. Then M is called a recurrent space, if its curvature tensor

$$B^i_{jkl}=\Gamma^i_{jk,l}-\Gamma^i_{jl,k}+\Gamma^h_{jk}\Gamma^i_{hl}-\Gamma^h_{jl}\Gamma^i_{hk}$$

satisfies the relations

$$(1.1) \qquad B^i_{jkl;m}=B^i_{jkl}K_m,$$

where a comma denotes the ordinary differentiation and a semi-colon denotes covariant differentiation with respect to the Γ^i_{jk}.

An infinitesimal transformation v^k is called a conformal collineation if it satisfies the equations

$$(1.2) \qquad \pounds\Gamma^i_{jk}=v^i{}_{;j;k}+v^l B^i_{jkl}=\delta^i_j\sigma_k+\delta^i_k\sigma_j-g_{jk}\sigma^i,$$

where £ denotes the Lie differentiation with respect to v^k and σ_k is a vector field. Here we can easily see that the vector σ_i is the gradient vector field of a scalar function σ. This conformal collineation is a generalization of a conformal transformation and it becomes an conformal transformation if M is irreducible. [1]

Substituting (1.2) into the well known formulae [2]

$$(1.3) \qquad \pounds B^i_{jkl}=\left(\pounds\Gamma^i_{jk}\right)_{;l}-\left(\pounds\Gamma^i_{jl}\right)_{;k},$$

we obtain the equations

$$(1.4) \qquad \pounds B^i_{jkl}=v^a B^i_{jkl;a}-v^i{}_{;a}B^a_{jkl}+v^a{}_{;j}B^i_{akl}+v^a{}_{;k}B^i_{jal}+v^a{}_{;k}B_{jka}$$

$$=\delta^i_k\sigma_{j;l}-\delta^i_l\sigma_{j;k}+g_{jl}\sigma^i{}_{;k}-g_{jk}\sigma^i{}_{;l}.$$

If recurrent space M admits a conformal collineation

$$(1.5) \qquad \bar{x}^i=x^i+v^i(x)dt, \quad v^i{}_{;j}=p(x)\delta^i_j,$$

we have following equations by means of (1.1) and (1.4)

$$(1.6) \qquad (v^a K_a+2p)B^i_{jkl}=\delta^i_k\sigma_{j;l}-\delta^i_l\sigma_{j;k}+g_{jl}\sigma^i{}_{;k}-g_{jk}\sigma^i{}_{;l}.$$

And contracting (1.6) with respect to i and l and multiplying it by g_{ik}, we have

(1.7) $$P=\frac{1-n}{B}\sigma^a{}_{;a}-\frac{1}{2}v^aK_a,\quad (B\neq 0).$$

Hence the conformal collineation (1.5) takes the forms

(1.8) $$\bar{x}^i=x^i+v^i(x)dt,\quad v^i{}_{;j}=\left(\frac{1-n}{B}\sigma^a{}_{;a}-\frac{1}{2}v^aK_a\right)\delta^i_j.$$

Substituting the later of (1.8) into the equations

(1.9) $$\pounds\Gamma^i_{jk}=v^i{}_{;j;k}+B^i_{jka}v^a,$$

we have

(1.10) $$B^i_{jka}v^a=\delta^i_k\sigma_j-g_{jk}\sigma^i-\Big(\frac{n-1}{B^2}B_{;k}\sigma^a{}_{;a}-\frac{n-1}{B}\sigma^a{}_{;a;k}$$
$$-\frac{1}{2}v^a{}_{;k}K_a-\frac{1}{2}v^aK_{a;k}-\sigma_k\Big)\delta^i_j,$$

and substituting (1.1) into the Bianchi's identities

$$B^i_{jkl;m}+B^i_{jlm;k}+B^i_{jmk;l}=0,$$

we get

$$B^i_{jkl}K_m+B^i_{jlm}K_k+B^i_{jmk}K_l=0.$$

Multiplying these identities by v^l and summing for l, we have

(1.11) $$B^i_{jmk}v^aK_a=-B^i_{jka}v^aK_m+B^i_{jma}v^aK_k,$$

and subsituting (1.10) into the right hand side of (1.11) we have

(1.12) $$B^i_{jmk}v^aK_a=\left(\delta^i_mK_k-\delta^i_kK_m\right)\sigma_j-\left(g_{jm}K_k-g_{jk}K_m\right)\sigma^i$$
$$-\frac{n-1}{B^2}\left(B_mK_k-B_kK_m\right)\sigma^a{}_{;a}\delta^i_j+\frac{n-1}{B}\left(\sigma^a{}_{;a;m}K_k-\sigma^a{}_{;a;k}K_m\right)\delta^i_j$$
$$+\frac{1}{2}v^a\left(K_{a;m}K_k-K_{a;k}K_m\right)\delta^i_j+\left(\sigma_mK_k-\sigma_kK_m\right)\delta^i_j.$$

Substituting (1.12) into the equations

$$B^i_{jmk}v^aK_a+B^i_{mkj}v^aK_a+B^i_{kjm}v^aK_a=0,$$

and contracting with respect to i and j, we have

$$\frac{n-1}{B^2}\left(B_kK_m-B_mK_k\right)\sigma^a{}_{;a}+\frac{n-1}{B}\left(\sigma^a{}_{;a;m}K_k-\sigma^a{}_{;a;k}K_m\right)$$
$$+\frac{1}{2}v^a\left(K_{a;m}K_k-K_{a;k}K_m\right)+\left(K_m\sigma_k-K_k\sigma_m\right)=0,$$

where $n>2$.

Substituting above equations into (1.12), we have

$$(1.13)\qquad B^i_{jmk}v^aK_a=\left(\delta^i_mK_k-\delta^i_kK_m\right)\sigma_j-\left(g_{jm}K_k-g_{jk}K_m\right)\sigma^i$$
$$+2\left(\sigma_mK_k-\sigma_kK_m\right)\delta^i_j$$

Contracting (1.13) with respect to i and k and multiplying it by g_{jm}, we obtain

$$Bv^aK_a=2(1-n)K_a\sigma^a\ ,$$

and hence, we have by means of (1.7)

$$P=\frac{1-n}{B}\left(\sigma^a{}_{;a}-K_a\sigma^a\right)=\frac{1-n}{B^2}\left(B\sigma^a{}_{;a}-B_a\sigma^a\right).$$

Making use of (1.1), above equations can be written in the form

$$(1.14)\qquad P=(1-n)(\sigma^a/B)_{;a},$$

and hence we have the following:

THEOREM 1. *When a recurrent space M (Dim. of M>2) with non-zero scalar curvatnre admits an infinitesimal conformal collineation of the form* (1.5), *a necessary and sufficient condition that the vector* $v^i(x)$ *spans a contrafield in M is* $(\sigma^a/B)_{;a}=0$.

§2. Two recurrent spaces in projective correspondence.

Let M be a recurrent space with connection Γ^i_{jk}, of M and $\overline{M}$ are in projective correspondence, the connections are related by

$$(2.1)\qquad \overline{\Gamma}^i_{jk}=\Gamma^i_{jk}+\delta^i_j\varphi_k+\delta^i_k\varphi_j$$

For an arbitrary tensor A^{ij}, we have

$$A^{ij}{}_{|l}=A^{ij}{}_{;l}+2A^{ij}\varphi_l+\delta^i_l\varphi_aA^{aj}+\delta^j_l\varphi_aA^{ia},$$

and for an arbitrary tensor A_{ij}, we have

$$A_{ij|l}=A_{ij;l}-2A_{ij}\varphi_l-A_{lj}\varphi_i-A_{il}\varphi_j,$$

where the solidus ($|$) denotes the covariant derivative with respect to $\overline{\Gamma}^i_{jk}$. In general, we have the formulae

$$(2.2)\qquad T^{ab\cdots\cdots c}_{pq\cdots\cdots r|l}=T^{ab\cdots\cdots c}_{pq\cdots\cdots r;l}+(p-q)T^{ab\cdots\cdots c}_{pq\cdots\cdots r}\varphi_l$$
$$+\delta^a_l\varphi_mT^{mb\cdots\cdots c}_{pq\cdots\cdots r}+\cdots\cdots\cdots+\delta^c_l\varphi_mT^{ab\cdots\cdots m}_{pq\cdots\cdots r}$$

$$-\varphi_p T^{ab\cdots\cdots c}_{eq\cdots\cdots r} - \cdots\cdots\cdots - \varphi_r T^{ab\cdots\cdots m}_{pq\cdots\cdots l}$$

Using of above formulae (2.2), we have

$$(2.3) \qquad v^i{}_{|j|k} = v^i{}_{;j;k} + v^i\varphi_{j;k} + \delta^i_k \varphi_a v^a{}_{;j} + \delta^i_j \left(\varphi_a v^a\right)_{;k} + \delta^i_k \left(\varphi_a v^a\right)\varphi_j - v^i \varphi_j \varphi_k.$$

Making use of (2.1), we have

$$(2.4) \qquad \overline{B}^i_{jkl} = B^i_{jkl} + \delta^i_k \varphi_{j;l} - \delta^i_l \varphi_{j;k} + \varphi_j\left(\delta^i_l \varphi_k - \delta^i_k \varphi_l\right).$$

Substituting (2.3) and (2.4) into the equations

$$£\overline{\Gamma}^i_{jk} = v^i{}_{;j;k} + \overline{B}^i_{jkl} v^l ,$$

and remaking φ_a is a gradient vector, we have

$$(2.5) \qquad £\overline{\Gamma}^i_{jk} = £\Gamma^i_{jk} + \delta^i_j\left(\varphi_a v^a\right)_{;k} + \delta^i_k\left(\varphi_a v^a\right)_{;k}.$$

Hence we have the following:

THEOREM 2. *If two Riemannian spaces are in projective correspondence. then an affine motion of a space is a projective motion of another space.*

Multiplying g_{im} to the equations (1.2), we have

$$v_{m;j;k} + B_{mjkl}\ v^l = g_{jm}\sigma_k + g_{km}\sigma_j - g_{jk}\sigma_m.$$

Adding the equalions obtaioned by changing m and j in above equations, we have

$$(2.6) \qquad \sigma_k = \frac{1}{n} v^i{}_{;i;k}.$$

If the conformal collineation (1.2) admits the forms

$$(2.7) \qquad \overline{x}^i = x^i + v^i(x)dt, \qquad v^i{}_{;j} = \delta^i_j,$$

then, by means of the equations (2.6), σ_k vanish.

If the infinitesimal transformation (2.7) is also a conformal collineation in M, by equations (1.6) and (2.4), the function φ in (2.1) is characterized by the following equations:

$$(2.8) \qquad \varphi_{j;l} = \varphi_j \varphi_l.$$

Hence we have the following:

THEOREM 3. *If an infinitesimal conformal collineation of the form* (2.7) *is also a conformal collineation in M, the funition φ defining the projective correspondence between M and $\overline{M}$ satisfies the equation* (2.8).

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REFERENCES

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