ON THE CONTINUITY OF GROUP OPERATIONS OF AN L-GROUP WITH CP-IDEAL TOPOLOGY

By Tae Ho Choe

1. Introduction Recently, T. Naito [1] has introduced the concepts of *P*-*CP*-, *MP*-ideal topology in a lattice. And he has proved the following theorem ([1] Theorem 10 in Chapter III): In any commutative l-group, the group operations are continuous with respect to its *CP*-ideal topology.

In this short paper, we shall find a sufficient condition that the group operations of a noncommutative l-group are continuous with respect to its *CP*-ideal topology.

2. Definitions and preliminaries. Let L be a lattice. A subset I of a lattice L is called to be a *prime ideal* if and only if the following conditions hold:

(i) $x \leqq y$ and $y \in I$ imply $x \in I$.

(ii) $x \in I$ and $y \in I$ imply $x \cup y \in I$.

(iii) $x \cap y \in I$ implies $x \in I$ or $y \in I$.

And a prime ideal I is called to be a *CP-ideal* if and only if the following condition holds:

(iv) if $\{x_\alpha \mid \alpha \in \Delta\} \subseteq I$, and there exists $\sup_{\alpha \in \Delta} x_\alpha$, then $\sup_{\alpha \in \Delta} x_\alpha \in I$. The family of all *CP*-ideals is said to be a *CP-family*.

Dually, we can define the concepts of a dual prime ideal, a dual *CP*-ideal, and dual *CP*-family. And we shall denote by $C\mathfrak{P}$ the union of the *CP*-family and the dual *CP*-family. The *CP-ideal topology* of lattice L is that which results from taking $C\mathfrak{P}$ as a subbasis for the closed sets of the space L.

An l-group L is (i) a lattice (ii) a group, in which (iii) the inclusine relation is invariant under all group-translations $x \to a+x+b$

The following Lemmas are proved by T. Naito [1].

LEMMA 1. *In any l-group, if I is an element of $C\mathfrak{P}$, then each of $I+a$ and $-I$ is an element of $C\mathfrak{P}$.*

LEMMA 2. *In an l-group L with its CP-ideal topology,*
(a) any neighborhood of an element a of L can be written in the form $U+a$, where U is a neighborhood of zero element 0 of L,

(b) $-U$ *is a neighborhood of 0.*

We shall use the notations and terminologies in [1].

3 Results. First, we shall prove the following Lemma

LEMMA 3. *Let L be an l-group, in which* $\inf_\alpha x_\alpha$ *and* $\inf_\alpha (x_\alpha+a+x_\alpha)$ *exist for some set* $\{x_\alpha \mid \alpha\in\Delta\}$ *in L. Then* $\inf_\alpha (x_\alpha+a+x_\alpha)=\inf_\alpha x_\alpha+a+\inf_\alpha x_\alpha$ *if and only if* $x_\alpha+a+x_\beta \geqq (x_\alpha+a+x_\alpha)\cap(x_\beta+a+x_\beta)$ *for any* $x_\alpha,\ x_\beta \in\{x_\alpha\mid\alpha\in\Delta\}$.

PROOF. Assume $\inf_\alpha (x_\alpha+a+x_\alpha)=\inf_\alpha x_\alpha+a+\inf_\alpha x_\alpha$. Then we have $(x_\alpha+a+x_\alpha)\cap(x_\beta+a+x_\beta)=(x_\alpha+a+x_\alpha)\cap(x_\alpha+a+x_\beta)\cap(x_\beta+a+x_\alpha)\cap(x_\beta+a+x_\beta)$ for any $x_\alpha,\ x_\beta\in\{x_\alpha\mid\alpha\in\Delta\}$. Hence $x_\alpha+a+x_\beta\geqq(x_\alpha+a+x_\alpha)\cap(x_\beta+a+x_\beta)$. Conversely, we put $\inf_\alpha x_\alpha=x$. Since $x_\alpha\geqq x$ for all $\alpha\in\Delta$, we have $x_\alpha+a+x_\alpha\geqq x+a+x$ for all $\alpha\in\Delta$. While $a+\inf_\alpha x_\alpha=\inf_\alpha(a+x_\alpha)$ in any l-group, we have $x+a+x=\inf\{\inf_{\alpha\ \beta}(x_\alpha+a+x_\beta)\}$, where $x=\inf_\alpha x_\alpha=\inf_\beta x_\beta$. By the hypothesis, $x_\alpha+a+x_\beta\geqq(x_\alpha+a+x_\alpha)\cap(x_\beta+a+x_\beta)$ for any x_α, x_β. From these facts, we see $x+a+x=\inf_\alpha(x_\alpha+a+x_\alpha)$.

Finally, we prove the following theorem

THEOREM. *Let L be an l-group, in which if* $\inf_\alpha x_\alpha$ *exists for some* $\{x_\alpha\mid\alpha\in\Delta\}$, *there exists* $\inf_\alpha(x_\alpha+a+x_\alpha)$, *moreover* $\inf_\alpha(x_\alpha+a+x_\alpha)=\inf_\alpha x_\alpha+a+\inf_\alpha x_\alpha$, *and dual. Then the group operations of L are continuous with respect to its CP-ideal topology.*

PROOF. The following method of proof is due to T. Naito.
By Lemma 2, it is sufficient to show that for $U+(p-q)$, a neighborhood of $p-q$, there exist U_1+p, U_2+q such that

$$(U_1+p)-(U_2+q)\subseteq U+(p-q)\ :\ U_1+(p-q)-U_2\subseteq U+(p-q)$$

where U, U_1 and U_2 are neighborhoods of 0. Since $-U_2$ is a neighborhood of 0, we shall show that for any neighborhood $U+a$ of a there exists a neighborhood U_1 of 0 such that $U+a\supseteq U_1+a+U_1$, where $a=p+q$.
We shall devide into several cases.

Case 1). U^c (=complement of U in L) is a deal CP-ideal. Then U^c+a is also a dual CP-ideal by Lemma 1. Let I be the set of all x such that $x+a+x\in U^c+a$: $I=\{x\mid x+a+x\in U^c+a\}$. Then I is a dual CP-ideal. In fact, if $x\in I$ and $x\leqq y$, then $x+a+x\in U^c+a$ and $x+a+x\leqq y+a+y$. Hence we have $y+a+y\in U^c+a$: $y\in I$. If $x_\alpha\in I$ and $\inf_\alpha x_\alpha=x$, then $x_\alpha+a+x_\alpha\in U^c+a$ for all $\alpha\in\Delta$. By the hypothesis

$\inf_\alpha(x_\alpha+a+x_\alpha)$ exists and $\inf_\alpha(x_\alpha+a+x_\alpha)=x+a+x\in U^c+a$. Hence we have $x\in I$. If $x\cup y\in I$, then $(x\cup y)+a+(x\cup y)\in U^c+a$. By the duality of hypothesis, we see $(x\cup y)+a+(x\cup y)=(x+a+x)\cup(y+a+y)\in U^c+a$. This means that $x+a+x$ or $y+a+y\in U^c+a$, i.e. x or $y\in I$. Hence I is a dual CP-ideal of L. Now we put $I^c=U_1$, then U_1 is a neighborhood of 0, because $a\notin U^c+a$. Now we shall prove $U_1+a+U_1\subseteq U+a$. If $x\in U_1$, $y\in U_1$, then $x+a+x$, $y+a+y\notin U^c+a=(U+a)^c$, i.e. $x+a+x$, $y+a+y\in U+a$. By the hypothesis and Lemma 3, we see easily that $x+a+y\leqq(x+a+x)\cup(y+a+y)$. And we have $x+a+y\in U+a$. In fact, if $x+a+y\in U^c+a$, then $x+a+x$ or $y+a+y\in U^c+a$ which is contrary. Hence $U_1+a+U_1\subseteq U+a$.

Case 2) U^c is a CP-ideal. This case is dual of case 1).

Case 3) U is any neighborhood of 0. U^c can be written in the form

$$U^c=\bigwedge_{\alpha}\bigvee_{\beta=1}^{n_\alpha}I_{\alpha\beta}:U=\bigvee_{\alpha}\bigwedge_{\beta=1}^{n_\alpha}I^c_{\alpha\beta}.$$

Since U is a neighborhood of 0, there exists α_0 such that $\bigwedge_{\beta=1}^{n_\alpha}I^c_{\alpha_0\beta}\ni 0$. By case 1) and case 2) for each $I^c_{\alpha_0\beta}$, there exists a neighborhood $U_{1\beta}$ of 0 such that $U_{1\beta}+a+U_{1\beta}\subseteq I^c_{\alpha_0\beta}+a$. We put $U_1=\bigwedge_{\beta=1}^{n_\alpha}U_{1\beta}$ then U_1 is a neighborhood of 0 and $U_1+a+U_1\subseteq\bigwedge_{\beta=1}^{n_{\alpha_0}}(I^c_{\alpha_0\beta}+a)=(\bigwedge_{\beta=1}^{n_{\alpha_0}}I^c_{\alpha_0\beta})+a\subseteq U+a$. This completes the proof.

COROLLARY 1. *If L be a commutative l-group, then the group operations of L are continuous with respect to its CP-ideal topology.* (This theorem has already proved by T. Naito in the same way)

In fact, in any commutative l-group, the hypothesis in above Theorem is satisfying by Lemma 2.

COROLLARY 2. *In an l-group with continuous group operations with respect to its CP-ideal topology, if $x_\alpha\downarrow x$, then $(x_\alpha+a+x_\alpha)\downarrow(x+a+x)$, if $\inf_\alpha(x_\alpha+a+x_\alpha)$ exists.*

In fact, by Note in [1] (p. 241), if $x_\alpha\downarrow x$, then $x_\alpha\to x$ (CP-ideal topology). Hence, since group operations are continuous, we have $x_\alpha+a+x_\alpha\to x+a+x$ (CP-ideal topology). On the other hand, we have $(x_\alpha+a+x_\alpha)\downarrow\inf_\alpha(x_\alpha+x+x_\alpha)$. Therefore $x_\alpha+a+x_\alpha\to\inf_\alpha(x_\alpha+a+x_\alpha)$ (CP-ideal topology). Since L is a T_2-space, we have $\inf_\alpha(x_\alpha+a+x_\alpha)=x+a+x$, i.e. $x_\alpha+a+x_\alpha\downarrow x+a+x$.

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Mathematical Department

Liberal Arts and Science College
Kyungpook University.

REFERENCE

[1] T. Naito, *Lattice with P-ideal Topologies*, Tohoku. Math. Jour. Sendai (Japan) **12** (1960) pp. 235-262