THE INFINITESIMAL TRANSFORMATIONS IN THE PARAMETER GROUP MANIFOLDS

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Introduction.

The differential geometric properties of the parameter group manifolds were studied by Nobuo Horie and he introduced a metric and found Riemann Curvature. The purpose of present paper is to study on the infinitesimal transformations in these manifolds. In section 1, we shall take necessary results for this paper from N. Horie's paper [1]. In section 2 and 3, we shall define the infinitesimal fundamental transformation, which is defined so that the trajectories of one-parameter sub group are unchanged, and contrast this with infinitesimal affine collineation and projective transformation. In section 4, we shall moreover add the infinitesimal motion and contrast these infinitisimal transformations.

1. Preliminaries.

On a continuous transformation group with n independent variables and r essential parameters, of which the equations are defined by

$$x'^i=f^i(x,\ a)\quad (i=1,\ \cdots\cdots,\ n),$$

where a's are r parameters and x's independent variables, let us take the followings as the equations of the combinations in the group of dimension r;

$$a_3^\alpha=\varphi^\alpha(a_1, a_2)\quad (\alpha=1,\ \cdots\cdots,\ r).$$

Then we consider the equations

$$\frac{\partial a_3^\alpha}{\partial a_2^\beta}=A_b^\alpha(a_3)A_\beta^b(a_2),\quad \frac{\partial a_3^\alpha}{\partial a_1^\beta}=\bar{A}_b^\alpha(a_3)\bar{A}_\beta^b(a_1)$$

as the fundamental equations of parameter groups $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$, namely, the first and second parameter groups respectively, where the determinant $|A_b^\alpha|$ is not zero, $\|A_\alpha^b\|$ is the inverse matrix of $\|A_b^\alpha\|$ and $\bar{A}$'s are similar to A's. From this, we may define the group manifolds with connections

$$(1.\ 1)\qquad L_{\beta\gamma}^\alpha=A_b^\alpha\frac{\partial A_\beta^b}{\partial a^\gamma}=-A_\beta^b\frac{\partial A_b^\alpha}{\partial a^\gamma},$$

$$\bar{L}_{\beta\gamma}^\alpha=\bar{A}_b^\alpha\frac{\partial \bar{A}_\beta^b}{\partial a^\gamma}=-\bar{A}_\beta^b\frac{\partial \bar{A}_b^\alpha}{\partial a^\gamma}$$

and

(1. 2) $$\Gamma^{\alpha}_{\beta\gamma}=\frac{1}{2}(L^{\alpha}_{\beta\gamma}+\bar{L}^{\alpha}_{\beta\gamma})=\frac{1}{2}(L^{\alpha}_{\beta\gamma}+L^{\alpha}_{\gamma\beta}),$$

which are denoted by $\mathcal{G}^{(+)}$, $\mathcal{G}^{(-)}$ and $\mathcal{G}^{(0)}$. Moreover, in $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$, they have torsions, whose components are

(1. 3) $$\Omega^{\alpha}_{\beta\gamma}=\frac{1}{2}(L^{\alpha}_{\beta\gamma}-L^{\alpha}_{\gamma\beta})=\frac{1}{2}C^{e}_{bc}A^{\alpha}_{e}A^{b}_{\beta}A^{c}_{\gamma}$$

by virtue of Maurer-Cartan equations, where C^{e}_{bc} are constants of structure and skew symmetric with respect to b and c.

On $\mathcal{G}^{(+)}$, there is defined the metric tensor by

(1. 4) $$g_{\alpha\beta}=\sum_{a}A^{a}_{\alpha}A^{a}_{\beta}$$

and the Christoffel symbols are represented by

(1. 5) $$\left\{{\alpha \atop \beta\gamma}\right\}=\Gamma^{\alpha}_{\beta\gamma}+\frac{1}{2}(C^{e}_{ab}+C^{b}_{ae})A^{\alpha}_{a}A^{e}_{\beta}A^{b}_{\gamma}.$$

Thus, we denote $\mathcal{G}^{(+)}$ with metric (1. 4) by $\mathcal{G}^{(r)}$. And we shall denote the covariant differentiations of a tensor with respect to $L^{\alpha}_{\beta\gamma}$, $\Gamma^{\alpha}_{\beta\gamma}$ and $\left\{{\alpha \atop \beta\gamma}\right\}$ by "|", "," and ";" respectively, especially,

(1. 6) $$S^{\alpha}_{\beta|\gamma}=\frac{\partial S^{\alpha}_{\beta}}{\partial a^{\gamma}}+L^{\alpha}_{\gamma\tau}S^{\tau}_{\beta}-L^{\tau}_{\beta\gamma}S^{\alpha}_{\tau}.$$

Next, we can show that the curvature tensors on $\mathcal{G}^{(+)}$, $\mathcal{G}^{(-)}$, $\mathcal{G}^{(0)}$ and $\mathcal{G}^{(r)}$ are represented by respectively

(1. 7) $$L^{\alpha}_{\beta\gamma\delta}=0,\quad \bar{L}^{\alpha}_{\beta\gamma\delta}=0,$$

(1. 8) $$\Gamma^{\alpha}_{\beta\gamma\delta}=\frac{1}{4}C^{e}_{dc}C^{a}_{eb}A^{\alpha}_{a}A^{b}_{\beta}A^{c}_{\gamma}A^{d}_{\delta}$$

and

(1. 9) $$R^{\alpha}_{\beta\gamma\delta}=\frac{1}{4}C_{abcd}A^{\alpha}_{a}A^{b}_{\beta}A^{c}_{\gamma}A^{d}_{\delta},$$

(1. 10) $$\begin{aligned}C_{abcd}=&C^{e}_{dc}C^{a}_{eb}+C^{a}_{ed}(C^{b}_{ec}+C^{c}_{eb})+C^{a}_{ce}(C^{b}_{ed}+C^{d}_{eb})+C^{e}_{bc}(C^{e}_{ad}+C^{d}_{ae})\\&+C^{e}_{db}(C^{e}_{ac}+C^{c}_{ae})+2C^{e}_{dc}(C^{b}_{ae}+C^{e}_{ab})+(C^{b}_{ed}+C^{d}_{eb})(C^{e}_{ac}+C^{c}_{ae})\\&+(C^{c}_{be}+C^{e}_{bc})(C^{e}_{ad}+C^{d}_{ae}).\end{aligned}$$

2. The infinitesimal transformations on $\mathcal{G}^{(+)}$ with connection $L^{\alpha}_{\beta\gamma}$.

In $\mathcal{G}^{(+)}$, the trajectories of one-parameter sub group is expressed by

(2. 1) $$\frac{da^{\alpha}}{dt}=e^{a}A^{\alpha}_{a}(a),$$

where e^a is constants and one of them at least is not zero. And let us assume that there exist the r-independent sub groups.

If a transformation

(2. 2) $$a'^{\alpha}=a^{\alpha}+\xi^{\alpha}\delta t$$

earies, infinitesimally, every trajectory of (2. 1) in $\mathcal{G}^{(+)}$ into a trajectory (2. 1), it is called by an *infinitesimal fundamental transformation.*

Now, if the transformation (2. 2) is an infinitesimal fundamental transformation, then it will carry the one-parameter sub group (2. 1) into the one parameter sub group

(2. 3) $$\frac{da'^{\alpha}}{dt}=e^{a}A_{a}^{\alpha}(a'),$$

where we take parameter t commonly. From (2. 3) we have

$$\left(\delta_{\beta}^{\alpha}+\frac{\partial\xi^{\alpha}}{\partial a^{\beta}}\delta t\right)\frac{da^{\beta}}{dt}=e^{a}\left(A_{a}^{\alpha}+\frac{\partial A_{a}^{\alpha}}{\partial a^{\beta}}\xi^{\beta}\delta t\right),$$

from which, substituting (2. 1) in this, we obtain

$$e^{a}\left(\frac{\partial\xi^{\alpha}}{\partial a^{\beta}}A_{\alpha}^{\beta}-\frac{\partial A_{a}^{\alpha}}{\partial a^{\beta}}\right)\delta t=0.$$

Since that there exist r-independent sub groups, we must have

(2. 4) $$\frac{\partial\xi^{\alpha}}{\partial a^{\beta}}A_{a}^{\beta}-\frac{\partial A_{a}^{\alpha}}{\partial a^{\beta}}\xi^{\beta}=0,$$

or, in tensor form,

$$A_{a}^{\tau}\xi^{\alpha}{}_{|\tau}=0,$$

or

(2. 5) $$\xi^{\alpha}{}_{|\tau}=0.$$

On the other hand, if we calculate the Lie derivative of A_{a}^{α}, we have

$$\pounds A_{a}^{\alpha}=\frac{A_{a}^{\alpha}(a')-A'^{\alpha}_{a}(a')}{\delta t}=\frac{\partial A_{a}^{\alpha}}{\partial a^{\tau}}\xi^{\tau}-A_{\alpha}^{\tau}\frac{\partial\xi^{\alpha}}{\partial a^{\tau}},$$

or

(2. 6) $$\pounds A_{a}^{\alpha}=-A_{a}^{\tau}\xi^{\alpha}{}_{|\tau},$$

and similarly, we have

(2. 7) $$\pounds A_{\alpha}^{a}=A_{\tau}^{a}\xi^{\tau}{}_{|\alpha},$$

and from (2. 5), we obtain $\pounds A_{\alpha}^{a}=0$ and $\pounds A_{a}^{\alpha}=0$. In consequence of that $\pounds(L_{\beta\gamma}^{\alpha})=\xi^{\alpha}{}_{|\beta|\gamma}+L_{\beta\gamma\tau}^{\alpha}\xi^{\tau}$ [2] and (1. 7), we have

(2. 8) $$\pounds(L_{\beta\gamma}^{\alpha})=\xi^{\alpha}{}_{|\beta|\gamma}$$

and, from (2. 5),

$$\pounds(L^{\alpha}_{\beta\gamma})=0.$$

Hence, we have the followings:

THEOREM 1. *For the parameter group manifold* $\mathcal{G}^{(+)}$ *with connection* $L^{\alpha}_{\beta\gamma}$ *to admit an infinitesimal fundamntal transformation, it is necessary and sufficient that it holds* $\pounds A^{\alpha}_{a}=\pounds A^{a}_{\alpha}=0$.

COROLLARY. *If* $\mathcal{G}^{(+)}$ *with connection* $L^{\alpha}_{\beta\gamma}$ *admits an infinitesimal fundamental transformation, then the transformation is an infinitesimal afine collineation with respect to* $L^{\alpha}_{\beta\gamma}$.

An infinitesimal transformation (2. 2) is said to define an infinitesimal projective transformation on $\mathcal{G}^{(+)}$ with connection $L^{\alpha}_{\beta\gamma}$, if it satisfies

$$(2.\ 9)\qquad \pounds(L^{\alpha}_{\beta\gamma})=L^{\alpha}_{\beta\gamma}+\delta^{\alpha}_{\beta}\varphi_{\gamma}+\delta^{\alpha}_{\gamma}\varphi_{\beta}.$$

From (2. 8), we have

$$\xi^{\alpha}{}_{|\beta|\gamma}=L^{\alpha}_{\beta\gamma}+\delta^{\alpha}_{\beta}\varphi_{\gamma}+\delta^{\alpha}_{\gamma}\varphi_{\beta}.$$

If $\mathcal{G}^{(+)}$ admits an infinitesimal fundamental transformation, this is reduced into

$$L^{\alpha}_{\beta\gamma}+\delta^{\alpha}_{\beta}\varphi_{\gamma}+\delta^{\alpha}_{\gamma}\varphi_{\beta}=0,$$

since $\xi^{\alpha}{}_{|\beta}=0$. Contracting with respect to α and β, α and γ, and eliminating φ_{γ}, we obtain $L^{\alpha}_{\alpha\gamma}-L^{\alpha}_{\gamma\alpha}=0$.

From (1. 3), we have

$$C^{b}_{mn}A^{\alpha}_{b}A^{m}_{\alpha}A^{n}_{\gamma}=0,$$

or

$$(2.10)\qquad C^{m}_{mn}=0.$$

Hence we have the following:

THEOREM 2. *In* $\mathcal{G}^{(+)}$ *with connection* $L^{\alpha}_{\beta\gamma}$, *if an infinitesimal projective tranformation admits the infinitesimal fundamental transformation, then it satisfies* $C^{m}_{mn}=0$

From (2. 8) and Ricci formula with respect to connection $L^{\alpha}_{\beta\gamma}$, we have, since $L^{\alpha}_{\beta\gamma\sigma}=0$,

$$\pounds\Omega^{\alpha}_{\beta\gamma}=\xi^{\alpha}{}_{|\beta|\gamma}-\xi^{\alpha}{}_{|\gamma|\beta}=-2\xi^{\alpha}{}_{|\tau}\Omega^{\tau}_{\beta\gamma}.$$

From (1. 3) and (2. 6), we obtain

$$\pounds\Omega^{\alpha}_{\beta\gamma}=-C^{m}_{bc}A^{b}_{\beta}A^{c}_{\gamma}\pounds(A^{\sigma}_{m}),$$

and consequently, we may assert that $\pounds\Omega^{\alpha}_{\beta\gamma}=0$ and $C^{m}_{bc}\pounds(A^{\sigma}_{m})=0$ are equivalent.

Hence we have the following:

THEOREM 3. *In* $\mathscr{G}^{(+)}$ *with connection* $L^{\alpha}_{\beta\gamma}$, $\pounds\Omega^{\alpha}_{\beta\gamma}=0$ *and* $|\pounds(A^{\sigma}_{m})|=0$ *are equivalent provided that one of* $C^{1}_{bc}, \cdots\cdots, C^{r}_{bc}$ *at leat is not zero.*

3. The infinitesimal transformations on $\mathscr{G}^{(0)}$ with connection $\Gamma^{\alpha}_{\beta\gamma}$.

Let us consider the parameter group manifold $\mathscr{G}^{(0)}$ with connection $\Gamma^{\alpha}_{\beta\gamma}$. From (2. 6) and (2. 7), we have

$$\pounds A^{\alpha}_{a}=A^{\alpha}_{a,\tau}\,\xi^{\tau}-A^{\tau}_{a}\,\xi^{\alpha}{}_{,\tau}, \tag{3. 1}$$
$$\pounds A^{a}_{\alpha}=A^{a}_{\alpha,\tau}\,\xi^{\tau}-A^{a}_{\tau}\,\xi^{\tau}{}_{,\alpha}.$$

From (1. 1), (1. 2) and (1. 3), we have

$$A^{\tau}_{b,\beta}=\Omega^{\tau}_{\beta\lambda}A^{\lambda}_{b}=\frac{1}{2}C^{e}_{ab}A^{a}_{\beta}A^{\tau}_{e}, \tag{3. 2}$$
$$A^{e}_{\alpha,\beta}=\Omega^{\tau}_{\alpha\beta}A^{e}_{\tau}=-\frac{1}{2}C^{e}_{ab}A^{a}_{\alpha}A^{b}_{\beta}.$$

If we assume that $\mathscr{G}^{(0)}$ admits an infinitesimal fundamental transformation, from (3. 1), we obtain by virtue of (3. 2)

$$\xi^{\alpha}{}_{,\beta}=\xi^{\tau}A^{\alpha}_{a,\tau}A^{a}_{\beta}=\frac{1}{2}C^{a}_{bc}A^{\alpha}_{a}A^{b}_{\tau}A^{c}_{\beta}\xi^{\tau}. \tag{3. 3}$$

Differentiating (3. 3) by a^{γ}, substituting (3. 2) and (3. 3) in it and using of Jacobian identities of constants of structure, we have

$$\xi^{\alpha}{}_{,\beta,\gamma}=\frac{1}{4}C^{a}_{mb}C^{m}_{ec}A^{\alpha}_{a}A^{b}_{\beta}A^{c}_{\gamma}A^{e}_{\tau}\xi^{\tau}. \tag{3. 4}$$

Since $\pounds(\Gamma^{\alpha}_{\beta\gamma})=\xi^{\alpha}{}_{,\beta,\gamma}+\Gamma^{\alpha}_{\beta\gamma\tau}\xi^{\tau}$, from (1. 8) and (3. 4), we have

$$\pounds(\Gamma^{\alpha}_{\beta\gamma})=\frac{1}{2}C^{\alpha}_{mb}C^{m}_{ec}A^{\alpha}_{a}A^{b}_{\beta}A^{e}_{\gamma}A^{e}_{\tau}\,\xi^{\tau}. \tag{3. 5}$$

Hence we have the following:

THEOREM 4. *In* $\mathscr{G}^{(0)}$ *with connection* $\Gamma^{\alpha}_{\beta\gamma}$, *for an infinitesimal fundamental transformation to admit an infinitesimal affine collineation with respect to* $\Gamma^{\alpha}_{\beta\gamma}$, *it is necessary and sufficient that it holds*

$$C^{a}_{mb}C^{m}_{ec}A^{e}_{\tau}\,\xi^{\tau}=0. \tag{3. 6}$$

In $\mathscr{G}^{(0)}$ with connection $\Gamma^{\alpha}_{\beta\gamma}$, if an infinitesimal fundamental transformation admits the infinitesimal projective transformation, from (3. 5), we have

$$\frac{1}{2}C^a_{mb}C^m_{ec}A^\alpha_a A^b_\beta A^c_\gamma A^e_\tau \xi^\tau = \Gamma^\alpha_{\beta\gamma} + \delta^\alpha_\beta \varphi_\gamma + \delta^\alpha_\gamma \varphi_\beta.$$

Contracting with respect to α and β, α and γ, and eliminating φ_γ, we obtain

$$(C^a_{ma}C^m_{ec} - C^a_{mc}C^m_{ea})A^c_\gamma A^e_\tau \xi^\tau = 0.$$

Using of Jacobian identities, we have

$$(3.\ 7) \qquad C^a_{mc}C^m_{ac}A^e_\tau \xi^\tau = C^a_{ma}C^m_{ec}A^e_\tau \xi^\tau = 0.$$

THEOREM 5. *In* $\mathcal{G}^{(0)}$ *with connection with* $\Gamma^\alpha_{\beta\gamma}$, *if an infinitesimal fundamental transformation admits the infinitesimal projective transformation, then it satisfies* (3. 7).

4. The infinitesimal transformations in $\mathcal{G}^{(r)}$ with metric $g_{\alpha\beta} = \sum_a A^a_\alpha A^a_\beta$.

Let us consider the parameter group manifold $\mathcal{G}^{(r)}$ with metric $g_{\alpha\beta} = \sum_a A^a_\alpha A^a_\beta$. From (2. 6) and (2. 7), we have

$$(4.\ 1) \qquad \pounds A^\alpha_a = \xi^\tau A^\alpha_{a;\tau} - \xi^\alpha{}_{;\tau} A^\tau_a,$$

$$\pounds A^a_\alpha = \xi^\tau A^a_{\alpha;\tau} - \xi^\tau{}_{;\alpha} A^a_\tau.$$

From (1. 1), (1. 2), (1. 3) and (1. 5), we can calculate

$$(4.\ 2) \qquad A^\alpha_{a;\beta} = -\frac{1}{2}(C^e_{ma} + C^m_{ea} + C^a_{em})A^\alpha_e A^m_\beta,$$

$$A^a_{\alpha;\beta} = \frac{1}{2}(C^a_{me} + C^m_{ea} + C^e_{ma})A^m_\alpha A^e_\beta.$$

If $\mathcal{G}^{(r)}$ admits an infnitesimal fundamental transformation, we have

$$(4.\ 3) \qquad \pounds g_{\alpha\beta} \equiv \xi_{\alpha;\beta} + \xi_{\beta;\alpha} = 0,$$

and consequently, that an infinitesimal fundametal transformation admits the infinitesimal motion, since $\pounds A^a_\alpha = 0$. Conversely, if $\mathcal{G}^{(r)}$ admits an infinitesimal motion, we have from (4. 3)

$$(4.\ 4) \qquad \xi^\alpha{}_{;\alpha} = 0.$$

On the other hand, from (4. 1), we obtain

$$(4.\ 5) \qquad \xi^\alpha{}_{;\beta} = \xi^\tau A^\alpha_{a;\tau} A^a_\beta - A^a_\beta \pounds A^\alpha_a.$$

Using of (4. 2) and contracting with respect to α and β, we have, by virtue of (4. 4),

$$(4.\ 6) \qquad A^a_\alpha \pounds A^\alpha_a = 0, \quad A^\alpha_a \pounds A^a_\alpha = 0.$$

Hence we have the following:

THEOREM 6. *In* $\mathscr{G}^{(r)}$ *with metric* (1. 4), *an infinitesimal fundamental transformation admits the infinitesimal motion, and if* $\mathscr{G}^{(r)}$ *admits an infinitesimal motion, then it satisfies* (4. 6).

Since an infinitesimal motion admits the infinitesimal affine collineation, we have the followings:

COROLLARY. *In* $\mathscr{G}^{(r)}$ *with metric* (1. 4), *an infinitesimal fundamental transformation admits the infinitesimal affine collineation, that is, it holds*

$$(4.\ 7)\qquad L\{{}^{\alpha}_{\beta\gamma}\}\equiv\xi^{\alpha}{}_{;\beta;\gamma}+R^{\alpha}_{\beta\gamma\delta}\,\xi^{\delta}=0.$$

Differentiating (4. 5) with respect to a^{γ} covariantly, substituting (4. 2) and (4. 5) in it, we have

$$(8.\ 8)\qquad \xi^{\alpha}{}_{;\beta;\gamma}=\frac{1}{4}F_{acbd}A^{\alpha}_{a}A^{b}_{\beta}A^{c}_{\gamma}A^{d}_{\tau}\xi^{\tau},$$

where F_{acbd} is a expression formed the constants of structure. Adding F_{acbd} and C_{abcd} in (1. 10) and using of Jacobian identities, we have

$$(4.\ 9)\qquad K_{abcd}=F_{acbd}+C_{abcd},$$

and consequently,

$$(4.10)\qquad K_{abcd}\xi^{\tau}A^{m}_{\tau}=0.$$

Hence we have the following:

THEOREM 7. *If* $\mathscr{G}^{(r)}$ *with metric* (1. 4) *admits an infinitesimal fundamental transformation, it satisfies* (4. 10).

If an infinitesimal fundamental transformation admits the infinitesimal projective transformation in $\mathscr{G}^{(r)}$, then we have

$$K_{abcm}A^{\alpha}_{a}A^{b}_{\beta}A^{c}_{\gamma}A^{m}_{\tau}\,\xi^{\tau}=\{{}^{\alpha}_{\beta\gamma}\}+\delta^{\alpha}_{\beta}\varphi_{\gamma}+\delta^{\alpha}_{\gamma}\varphi_{\beta}.$$

Contracting with respective to α and β, α and γ, and eliminating φ_{γ}, we obtain

$$(4.11)\qquad (K_{aacm}-K_{acam})A^{m}_{\tau}\,\xi^{\tau}=0.$$

Calculating (4. 11) and using of Jacobian identities, we obtain

$$(4.12)\qquad [11C^{e}_{am}C^{e}_{at}+4C^{e}_{at}C^{a}_{em}+3C^{e}_{mt}C^{a}_{ac}+2(C^{m}_{te}C^{a}_{ac}$$
$$+C^{t}_{me}C^{a}_{ae}+C^{t}_{ea}C^{m}_{ae})+C^{e}_{am}C^{t}_{ae}+C^{e}_{ta}C^{m}_{ea}]A^{m}_{\tau}\,\xi^{\tau}=0.$$

Hence we have the following:

THEOREM 8. *In* $\mathcal{G}^{(n)}$ *with metric* (1. 4), *if an infinitesmal fundamental transformation admits the infinitesimal projective transformation, then it satisfies* (4. 12).

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