A NOTE ON CONVERGENCE CLASSES

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In the comment [I] I have given the necessary and sufficient conditions for the uniform structure by the convergence class. In this note I tried to prove the independency of the above conditions. Above conditions are as follows:

Let $\mathscr{L}=\{c_x\}$ be a family of orderedpairs (c, x), where C is a sequenc in a set X and x a point in X and Cx means (c, x).

(i) If $c=\{x_n\}$ is a sequence such that $x_n=x$ for each n, then $c_x\in\mathscr{L}$.

(ii) Relation $\geqq$ directs the set D and the range of a function N on $\mathscr{L}\times D$ $=\{(c_x, d)\}$ is the set of natural numbers, and if $d'\geqq d$, then $N(c_x, d')$ $\geqq N(c_x, d)$.

(iii) For each d in D and each x in X, there is a member d' in D such that if $x\in c_y(N(c_y, d))$ for c_y in $\mathscr{L}$, then $y\in c_x(N(c_x, d))$ for some c_x in $\mathscr{L}$.

(iv) For each d in D and each x in X there is a member d' in D such that if $x\in c_y(d')$ and $y\in c_z(d')$ for some c_y, c_x in $\mathscr{L}$ then $x\in c'_z(d)$ for some c'_z in $\mathscr{L}$. (where $c_x(d)=c_x(N(c_x, d))$ etc.)

In Lemma 1, we shall prove the independency of Kelley's five conditions: (a) Each members of $\mathcal{U}$ contains the diagonal Δ, (b) If $U\in\mathcal{U}$, then $U^{-1}\in\mathcal{U}$, (c) If $U\in\mathcal{U}$, then $V\circ V\subset U$ for some V in $\mathcal{U}$, (d) If U and V are members of $\mathcal{U}$, then $U\cap V\in\mathcal{U}$, and (e) If $U\in\mathcal{U}$ and $U\subset V\subset X\times X$ then $V\in\mathcal{U}$. And in theorem 1 the independency of the above four conditions i)—iv) for the convergence classes will be proved bassd on Lemma 1.

LEMMA 1. *The following conditions are mutually independent.*

(a) $U\in\mathcal{U}\longrightarrow U\supset\Delta$

(b) $U\in\mathcal{U}\longrightarrow U^{-1}\in\mathcal{U}$

(c) $U\in\mathcal{U}\longrightarrow\exists V\in\mathcal{U}:\quad V\circ V\subset U$

(d) $U, V\in\mathcal{U}\longrightarrow U\cap V\in\mathcal{U}$

(e) $U\in\mathcal{U}$ and $U\subset V\subset X\times X\longrightarrow V\in\mathcal{U}$.

PROOF.

(1) Independency of (a)

Let $X=\{a\}$ then $X\times X=\{(a, a)\}$.

Now let $\mathcal{U}=\{U_0, U_1\}$,

where $U_1 = \{(a,a)\}$, $U_0 = \{0\}$.
Then (b), (c), (d) and (e) are all satisfied, but (a) is not satisfied.

(2) Independency of (b).

Let $X = \{a, b\}$, then $X \times X = \{(a,a), (b,b), (a,b), (b,a)\}$.
Now let $U_1 = \{(a,a), (b,b), (a,b)\}$, $U_2 = \{(a,a), (b,b), (a,b), (b,a)\}$
and $\mathfrak{U} = \{U_1, U_2\}$, then
(c) is satisfied [since for all $U \in \mathfrak{U}$, $U_1 \circ U_1 \subseteq U$]
(d) is satisfied [clear]
(e) is satisfied [clear]
(b) is not satisfied.

(3) Independency of (c).

Let $X = \{a, b, c\}$ then
$X \times X = \{(a,a), (b,b), (c,c), (a,b), (b,a), (a,c), (c,a), (b,c), (c,b)\}$
New let $U_0 = \{(a,a), (b,b), (c,c), (a,b), (b,c)\}$

$U_1 = \{U_0, (a,c)\}$ $U_{10} = \{U_0, (c,a), (c,b)\}$
$U_2 = \{U_0, (c,a)\}$ $U_{11} = \{U_5, (b,a)\}$
$U_3 = \{U_0, (b,a)\}$. $U_{12} = \{U_5, (c,b)\}$
$U_4 = \{V_0, (c,b)\}$ $U_{13} = \{U_6, (c,b)\}$
$U_5 = \{U_0, (a,c), (c,a)\}$ $U_{14} = \{U_8, (c,b)\}$
$U_6 = \{U_0, (a,c), (b,a)\}$ $U_{15} = \{X \times X\}$
$U_7 = \{U_0, (a,c), (c,b)\}$ $U_{16+i} = U_i^{-1}$ $(i = 0, 1, 2, \cdots, 8)$
$U_8 = \{U_0, (b,a), (c,a)\}$ $U_{25+j} = U_{10+j}^{-1}$ $(j = 0, 1, 2)$
$U_9 = \{U_0(b,a), (c,b)\}$ $\mathfrak{U}_1, = \{U_0, U_1, \cdots, U_{15}\}$, $\mathfrak{U} = \{U_0, U_1, \cdots, U_{27}\}$

Then (a), (b), (d) and (e) are all satisfied but (c) is not satisfied since for U_0, there is no $U_i \in \mathfrak{U}$ such that $U_i \circ U_i \subset U_0$.

(4) Independency of (d).

Let $X = \{a, b\}$. $U_0 = \{(a,a), (b,b), (a,b)\}$
$U_1 = \{(a,a), (b,b), (b,a)\}$
$U_2 = \{(a,a), (b,b), (a,b), (b,a)\}$
and let $\mathfrak{U} = \{U_0, U_1, U_2\}$.
Then (a) and (b) are clearly satisfied.
For (c), i) $U_0 \circ U_0 \subset U_0$, ii) $U_1 \circ U_1 \subset U_1$, iii) $U_2 \circ U_2 \subset U_2$,
(d) is not satisfied since $U_0 \cap U_1 = \Delta \notin \mathfrak{U}$. (e) is clearly satisfied.

(5) Independency of (e).

Let $X=\{a, b\}$, and
let $U_0=\Delta$, $U_1=X\times X$.

And let $\mathfrak{U}=\{U_0, U_1\}$. Then (a), (b), (c) and (d) are clearly satisfied. But (e) is not satisfied.

THEOREM 1. *The four conditions* i)—iv) *for the convergence classes are mutually independent.*

PROOF. **(1) Independency of i).**

Let $X=\{a\}$ and let $\mathfrak{U}=\{U_0, U_1\}$ where $U_1=\{(a, a)\}$, $U_0=\{0\}$.

Let $\mathscr{L}$ be the null family of convergence sequences, then i) is not satisfied since $\{a, a, a, \cdots\cdots\}\notin\mathscr{L}$. In ii) let $d_0\geqq d_1$ where $d_0=U_0$, $d_1=U_1$. iii), iv) are clearly satisfied since the function N is not defined.

(2) Independency of ii).

Let $X=\{a, b\}$, and let $U_0=\{(a, a), (b, b)\}$, $U_1=\{(a, a), (b, b), (a, b)\}$, $U_2=\{U_0, (b, a)\}$, $U_3=\{U_1\cup U_2\}$ And let $U_i[x]$, where $i=0, 1, 2, 3$ and $x\in X$, be the neighborhoods of x. Let $\mathscr{L}$ be the class of all convergent sequences relative to the above neighborhood system. Let $D=\{d_0, d_1, d_2, d_3\}$ be the directed set with $d_0\geqq d_1\geqq d_3$, $d_0\geqq d_2\geqq d_3$. Then by Lemma 1, (c) of [1], the function N is defined such that $N(c_x, d_i)=N(c_x, U_i)$ with the exception of $N(c_a, d_0)=1$ and $N(c_b, d_0)=1$ where $c^a=\{a, b, a, a, \cdots\}$, $c_b=\{b, a, b, b, \cdots\}$ then $N(c_a, d_0)=1<N(c_a, d_2)=2$. Therefore ii) is not satisfied and conditions i), iii) and ix) are clearly satisfied.

(3) Independency of iii).

Let $X=\{a, b\}$ and $\mathfrak{U}=\{U_1, U_2\}$, where $U_1=\{(a, a), (b, b), (a, b)\}$, and let $\{a, b\}$ be the neighborhood of a and $\{b\}$ and $\{a, b\}$ be the neighborhoods of b. And let $\mathscr{L}$ be the class of all convergence sequences relative to the above neighborhoods system. And also let $d_1>d_2$ and $N(c_x, d_1)=N(c_x, U_1)$, $N(c_x, d_2)=N(c_x, U_2)$, then i), ii) are clearly satisfied. iii) is not satisfied since for $d_1\in D$ and $b\in X$, there is no d_i which satisfies the condition iii). iv) is clearly satisfied.

(4) Independency of iv).

Let $x=\{a, b, c\}$. In (c) of Lemma 1, $\mathfrak{U}_1=\{U_0, U_1, \cdots, U_{15}\}$ is directed by $\subset$. And let neighborhoods of the point a be $\{a, b\}$ and $\{a, b, c\}$.
Let neighborhoods of the point b be $\{b, c\}$ and $\{a, b, c\}$.

Let neighborhoods of the point c be $\{c\}$, $\{b, c\}$, $\{c, a\}$ and $\{a, b, c\}$.
Let $d_i \geqq d_j$ if $U_i \subset U_j$, then $D = \{d_i\}$ is directed by $\geqq$.

Let $\mathscr{C}$ be the class of all sequences c_x each of which converges to some point x in X relative to the above neighborhoods system, and let $N(c_x, d_i) = N(c_x, U_i)$ with the exception of $N(c_b, d_i) = 1$, $i = 0, 1, 2, 4, 5, 7, 10, 12$, and $N(c_c, d_j) = 1$, $j = 0, 1, 3, 6$ where $c_b = \{a, a, b, b, \cdots\}$ and $c_c = \{b, b, c, c, \cdots\}$. Then iv) is not satisfied since for $d_0 \epsilon D$ and the point $a \epsilon X$, there is no d_i which satisfies the condition iv). i) and ii) are clearly satisfied. iii) is also satisfied since for each point of X and each $d_i (i = 0, 1, \cdots, 15)$, there is d_0 which satisfies the condition iii).

Note: I wish to express here my hearty thanks to Prof. E. Kallin for his valuable remarks on my comment [1] who made them in Math. Review Vol 22, No. 1A (1961).

May 25, 1961

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REFERENCE

[1] C.Y.Kim *On definitions of a uniform space by the convergence class.* Kyungpook Math. Journal Vol.2, No. 1, (1959)