GAUSSIAN SPHERICAL REPRESENTATION OF A HYPERSURFACE OF AN EUCLIDEAN SPACE II

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1. Covariant derivative.

This paper is a continuation of [1]. In [1], we have studied some intrinsic properties of a Riemannian space S^n, which is the Gaussian spherical representation of a hypersurface V^n immersed in an (n+1)-dimensional Euclidean space E^{n+1} and the correspondence of the space S^n and V^n. Here, we are going to find some more relations between S^n and V^n by means of the covariant derivative and Lie derivative.

Let $g_{ij}(x)$ and $H_{ij}(x)$ be the first and second fundamental quantities of V^n respectively, then the Gauss and Codazzi equations are stated as follows:

(1. 1) $$R_{ijkl}=H_{ik}H_{jl}-H_{il}H_{jk},$$

(2. 2) $$H_{ij;k}-H_{ik;j}=0 .$$

Throughout this paper, the comma (,) and the semicolon (;) followed by indices denotes the ordinary and the covariant derivatives with respect to the Christoffel symbol of V^n, respectively.

If we denote the metric tensor of S^n by $C_{ij}(x)$ we have [1]

(1. 3) $$C_{ij}=g^{ab}H_{ai}H_{bj}.$$

By defining (H^{ab}) as the inverse matrix of (H_{ab}), we have

(1. 4) $$C^{ij}=g_{ab}H^{ai}H^{bj}.$$

Furthermore the Christoffel symbol π^i_{jk} and the curvature tensor Λ^i_{jkl} of S^n are given by

(1. 5) $$\pi^i_{jk}=H^{mi}(H_{mj,k}-\{{}^{a}_{mk}\}H_{aj})=H^{mi}(H_{mk,j}-\{{}^{a}_{mj}\}H_{ak}),$$

(1. 6) $$\Lambda^i_{jkl}=-H^{ai}H_{bj}R^{b}_{akl} .$$

For any contravariant vector V^i and covariant vector V_i, we have

(1. 7) $$V^i_{|l}=V^i_{;l}+H^{mi}H_{ma;l}V^a ,$$

(1. 8) $$V_{i|l}=V_{i;l}-H^{ma}H_{mi;l}V_a ,$$

where solidus (|) followed by indices denotes the covariant derivative with respect to π^{i}_{jk}.

In general, we can extend above process of covariant differentiation as follows:

(1. 9) $$\begin{aligned} T^{ab\cdots cd}{}_{\alpha\beta\cdots\gamma\delta|l} &= T^{ab\cdots cd}{}_{\alpha\beta\cdots\gamma\delta;l} \\ &+H_{ms;l}(H^{ma}T^{sb\cdots cd}{}_{\alpha\beta\cdots\gamma\delta}+\cdots+H^{md}T^{ab\cdots cs}{}_{\alpha\beta\cdots\gamma\delta}) \\ &-H^{ms}(H_{m\alpha;l}T^{ab\cdots cd}{}_{s\beta\cdots\gamma\delta}+\cdots+H_{m\delta;l}T^{ab\cdots cd}{}_{\alpha\beta\cdots\gamma s}), \end{aligned}$$

where $T^{ab\cdots cd}{}_{\alpha\beta\cdots\gamma\delta}$ is the components of any mixed tensor of contravariant p-order and covariant q-order.

As the particular case of the equation (1.9) we have followings:

(1.10) $$g^{ij}{}_{|l}=H_{ma;l}(H^{mi}g^{aj}+H^{mj}g^{ia}),$$

(1.11) $$g_{ij|l}=-H^{ma}(H_{mi;l}g_{aj}+H_{mj;l}g_{ia}),$$

(1.12) $$H_{ij|l}=-H_{ij;l} , \quad H^{ij}{}_{|l}=-H^{ij}{}_{;l} ,$$

(1.13) $$C_{ij|l}=0=C^{ij}{}_{|l} .$$

Making use of the Gauss equation (1.1) and (1.12) we have

(1.14) $$\Lambda^{i}_{jkl|s}=0 .$$

Hence we have

THEOREM 1. *The space S^n is a symmetric space for any Riemannian space V^n.*

2. A special homothetic transformation.

For an infinitesimal transformation (2.1) in V^n, the followrng equations hold [3]:

(2 .1) $$\bar{x}^i=x^i+\xi^i d\tau ,$$

(2. 2) $$£g_{jk}=\xi_{j;k}+\xi_{k;j} ,$$

(2 .3) $$£\{^{\,i}_{jk}\}=\xi^{i}{}_{;j;k}+R^{i}_{jks}\xi^{s} ,$$

(2. 4) $$£R^{i}_{jkl}=(£\{^{\,i}_{jk}\})_{;l}-(£\{^{\,i}_{jl}\})_{;k} ,$$

where the symbol £ denotes the Lie derivative.

In the space V^n, with positive definite quadratic differential form

$ds^2=g_{ij}dx^idx^j$, the infinitesimal transformation (2.1) is called conformal if the following relation holds [2]

$$(2.5)\qquad \pounds g_{ij}=\sigma g_{ij},$$

where σ is a function of x. The conformal transformation with the constant σ is called homothetic. And the necessary and sufficient condition that the infinitesimal transformation (2. 1) to be homothetic is analytically represented by the following equations .

$$(2.6)\qquad \pounds g_{jk}=\sigma g_{jk}, \qquad \pounds\{^{i}_{jk}\}=0 .$$

An infinitesimal transformation is called pseudo-homothetic if it is conformal transformation under which the curvature tensor is invariant, that is the following equations

$$(2.7)\qquad \pounds g_{jk}=\sigma g_{jk}, \qquad \pounds R^{i}_{jkl}=0 ,$$

are simultaneously satisfied.

From the equations (2. 2) and (1. 9) we have

$$(2.8)\qquad \pounds C_{jk}=\pounds g_{jk}-2H^{ma}H_{mj:k}\xi_a .$$

If the infinitesimal transformation (2. 1), which is conformal in V^n, is also a conformal transformation in S^n, then there exists a function $\psi(x)$ such that:

$$(2.9)\qquad \sigma g_{jk}-2H^{ma}H_{mj:k}\xi_a=\psi g^{ab}H_{aj}H_{bk} .$$

Multiplying g^{jk} to the equation (2. 9) and summing for j and k, we have

$$(2.10)\qquad n\sigma-2g^{jk}H^{ma}H_{mj:k}\xi_a=\psi g^{ab}g^{jk}H_{aj}H_{bk} .$$

By interchanging j and k in the equation (2.10) and subtracting it from (2.10), we have following equation by means of the equations of Gauss and Codazzi:

$$(2.11)\qquad \psi\cdot R=0 ,$$

where R is the scalar curvature of V^n, Hence we have

THEOREM 2. *The infinitesimal conformal transformation of a Riemannian space V^n with non vanishing scalar curvature is not an infinitesimal conformal transformation in S^n if ψ does not vanish.*

COROLLARY. *The infinitesimal conformal transformation of a Riemannian space V^n with non vanishing scalar curvature is a motion in S^n.*

THEOREM 3. *The homothetic transformation of a Riemannian space V^n with non vanishing scalar curvature is not a homothetic transfomation in S^n if ψ does not vanish.*

COROLLARY. *The homothetic transformation of a Riemannian space V^n with non vanishing scalar curvature is a moiton in S^n.*

After some more direct calculation, the Lie derivative of π^i_{jk} is given by

$$(2.12)\qquad £\pi^i_{jk}=£\{^i_{jk}\}+(\Lambda^i_{jka}-R^i_{jka})\xi^a+H^{mi}H_{ma;k}\ \xi^a_{;j} -H^{ma}H_{mj;k}\ \xi^i_{;a}+H^{mi}H_{ma;j}\xi^a_{;k}+H^{mi}H_{ma;j;k}\ \xi^a -H^{mi}H^{nb}H_{nj;k}\ H_{ma;b}\xi^a.$$

If the equation $H_{ij}=\varphi g_{ij}$ holds, the equations (2. 8) and (2.12) may be written as

$$(2.13)\qquad £C_{jk}=£g_{jk}-2\varphi^{-1}\varphi_{;k}\ \xi_j\ ,$$

$$(2.14)\qquad £\pi^i_{jk}=£\{^i_{jk}\}+\varphi^{-1}\ \varphi_{;j;k}\ \xi^i-\varphi^{-2}\varphi_{;j}\ \varphi_{;k}\ \xi^i+\varphi^{-1}\varphi_{;j}\ \xi^i_{;k}\ ,$$

where the function φ of x is not equal to zero. Moreover by the equation (1. 6) we have

$$(2.15)\qquad £\Lambda^i_{jkl}=£R^i_{jkl}\ .$$

Let us call the infinitesimal conformal transformation, the homothetic transformation, and the pseudo-homothetic transformation satisfying the equation $H_{ij}=\varphi g_{ij}$ in V^n "*special conformal*", "*special homothetic*" and "*special pseudo-homothetic*" respectively.

Now let us find out the condition that a special infinitesimal conformal transformation in V^n is also a conformal transformation in (2.16). If the conformal transformation $£g_{jk}=\sigma g_{jk}$ in V^n is conformal in Sn there exists a function ψ satisfying the following

$$(2.16)\qquad £C_{jk}=\psi C_{jk}\ .$$

By the equation (2.16) and (2.10), we have

$$(2.17)\qquad \psi=\sigma\varphi^{-2}-\frac{2}{n}\varphi^{-3}\varphi_{;k}\ \xi^k.$$

Conversely, if the equation (2.17) and the first equation of (2.6) are satisfied, the equation (2.16) should hold if the function φ satisfies the following equation:

(2.18) $$\varphi_{;a}\,\xi^a\,g_{jk}=n\varphi_{;k}\,\xi_j\,.$$

Hence we have

THEOREM 4. *If the equation* (2.17) *and* (2.18) *hold simultaneously, the special infinitesimal conformal transformation in* V^n *is a conformal one in* S^n.

By the equation (2.15) we have

THEOREM 5. *A special pseudo-homothetic transformation in* V^n *is a pseudo-homothetic transformation in* S^n.

For a special homothetic transformation in V^n, if it is a homothetic transformation in S^n, by the equations (2.14), (2.17) and (2.18) we have following equation

(2.19) $$\varphi_{,l}\,(\sigma-\frac{2}{n}\,\xi^k\,\partial_k\,log\,\varphi)=0$$

where $\partial_k\psi=\frac{\partial\psi}{\partial x^k}$. Hence we have

THEOREM 6. *If a special homothetic transformation in* V^n *is a homothetic one in* S^n, $\xi^k\,\partial_k\,log\,\varphi$ *is a constanti f* φ *is not a constant.*

We treat here another property of S^n. Let $\xi_{ab\cdots cd}$ be any harmonic tensor of V^n, the following equations hold [3]:

(2.20) $$\xi_{[ab\cdots cd;m]}=0\,,\quad g^{am}\,\xi_{ab\cdots cd;m}=0\,,$$

and $\xi_{ab\cdots cd}$ is skew symmetric in all the indices. From the equation (1.9), we have

(2.21) $$\xi_{ab\cdots cd|l}=\xi_{ab\cdots cd;l}-H^{mj}(H_{ma;l}\,\xi_{jb\cdots cd}+H_{mb;l}\,\xi_{aj\cdots cd}+\cdots+H_{md;l}\,\xi_{ab\cdots cj})\,.$$

By Codazzi relation (1.2) we have

(2.22) $$\xi_{[ab\cdots cd|m]}=0\,,$$

and $\xi^m{}_{b\cdots cd|m}=0$ if the following equation hold:

(2.23) $$H^{mj}\,\xi^{k}{}_{[b\cdots cd}H_{|mk|;j]}=0 .$$

Hence we have

THEOREM 7. *A necessary and sufficient condition that a harmonic tensor* $\xi_{ab\cdots cd}$ *in* V^n *to be a harmonic one in* S^n *is given by the equation* (2.23).

COROLLARY. *If we suppose the existence of the equation* $H_{ij}=\varphi_{ij}$ *and* φ *is not constant, then a harmonic tensor in* V^n *is also a harmonic one in* S^n.

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