LATTICE ORDERED COMMUTATIVE GROUPS OF THE SECOND KIND

By Tae Ho, Choe

Introduction. By a *partially (or lattice) ordered commutative group* (po. c. g. (or l. o. c. g.)) we mean a set G endowed with a binary operation $\cdot$ and a binary relation $\leqq$ such that the following axioms are satisfied

(i) G is a commutative group with respect to $\cdot$,

(ii) G is a partially ordered (or lattice) by $\leqq$,

(iii) if a and b are elements of G such that $a \leqq b$, then $ac \leqq bc$ for all c in G.

A. H. Clifford [1] has defined the concepts of *conserver* element and *inverter* element : in a totally ordered commutative semigroup, an element c is called conserver if $a<b$ implies $ca \leqq cb$, and an c inverter if $a<b$ implies $ca \geqq cb$. In this note, we define similar concepts as following. Let G satisfy just (i) and (ii) above. In G, an element c is *conserver* if $a \leqq b$, $a \# b$ (means a and b are incomparable) implies $ca \leqq cb$, $ca \# cb$, respectively, an *inverter* if $a<b$ (means $a \leqq b$ but $a \neq b$), $a \# b$ implies $ca > cb$. $ca \# cb$, respectively. G will be called a *partially (or lattice) ordered commutative group of the second kind* (=po. c. g. II(or l. o. c. g. II)) if G satisfies (i), (ii) and

(iv) Every element of G is either a conserver or an inverter or not both.

We call an element d *destroyer* if d is neither conserver nor inverter, i. e. if d is a destroyer, then there exists a pair of elements x and y in G such that $x<y$ and $dx \# dy$. We will give some typical examples of a destroyer in §4. And we shall call po. c. g. II G *simply ordered commutative group of the second kind* (s. o. c. g. II) if every element of G is either a consever or an inverter, and G simply ordered.

Let G_i $(i=1, 2, \cdots\cdots, n)$ be a s. o. c. g. II. By the *cardinal product* ΠG_i *of* G_i's we mean the set of all elements $(x_1, \cdots\cdots, x_n)$, x_i in G_i, where $(x_1, \cdots\cdots, x_n) \leqq (y_1, \cdots\cdots, y_n)$ if and only if $x_i \leqq y_i$ for all i. The class ΠG_i becomes a group if we define $(x_1, \cdots\cdots. x_n) \cdot (y_1, \cdots\cdots, y_n) = (x_1 y_1, \cdots\cdots, x_n y_n)$ for x_i, $y_i \in G_i$. Moreover, ΠG_i becomes a l. o. c. g. II. In §2, we deal some properties of l. o. c. g. II. And in §3, we shall investigate the necessary and sufficient conditions that a l. o. c. g. II is group-isomorphic to a cardinal product of some s. o. c. g. II's.

§2 **Some properties of l.o.c.g.II's** Let G be a l.o.c.g.II. Throughout this paper, A, B and D will denote the set of all conservers, inverters and destroyers, respectively. Evidently, $A^2 \subset A$, $AB \subset B$, $B^2 \subset A$, $A^{-1} \subset A$, $B^{-1} \subset B$, $AD \subset D$, $BD \subset D$, hence $D^{-1} \subset D$, where AB denotes the set of all elements ab, (a in A, b in B). Clearly, by definition, A, B and D are disjoint each other.

The following Lemmas are obvious.

[LEMMA 1] *A, $A \vee B$ are both subgroups of G.*

[LEMMA 2] *If $x \in A$ and $y \in B$, then*

(i) *$x(\alpha \cup \beta) = x\alpha \cup x\beta$ for any α, $\beta \in G$ and dually,*

(ii) *$y(\alpha \cup \beta) = y\alpha \cap y\beta$ for any α, $\beta \in G$ and dually.*

[LEMMA 3] *Let G be a l.o.c.g.II. A is a dual ideal (or ideal) of G if and only if $e < x$ (or $e > x$) implies $x \in A$. Where e is an identity of G.*

[PROOF] Assume A is a dual ideal. Then clearly, we see that $e < x$ implies $x \in A$. Conversely, assume $e < x$ implies $x \in A$. For $x \in A$ and $g \in G$, since $e < x \leqq x \cup g$, we have $x \cup g \in A$. And we have $a \cap b \in A$ for any $a, b \in A$. For, since $a \leqq a \cup b$, we have $a \cup b \in A$. Hence $(a \cup b)^{-1} \leqq a^{-1}, b^{-1}$. If $c \leqq a^{-1}, b^{-1}$, then we see $(a \cup b)^{-1} \cup c \leqq a^{-1}, b^{-1}$. Since $(a \cup b)^{-1} \cup c \in A$, $a \cup b \leqq [(a \cup b)^{-1} \cup c]^{-1}$, i.e. $(a \cup b)^{-1} \cup c \leqq (a \cup b)^{-1}$. Hence $c \leqq (a \cup b)^{-1}$, i.e. $(a \cup b)^{-1} = a^{-1} \cap b^{-1}$. Thus we have $a \cap b \in A$. i.e. A is a dual ideal, as desired.

$A < B$ means $a < b$ for all a in A, b in B

[THEOREM 1] *Let G be a l.o.c.g.II. Then*

(i) *A, B are anti-order isomorphic*

(ii) *D is the sum of some dA's, where d is a destroyer.*

(iii) *if $e < x$ implies $x \in A$, then A is an l-subgroup of G*

[PROOF] Let b be an elememt of G. Then $bA = B$. For, since $bA \subset BA \subset B$, we have $bA \subset B$. And $b_1 = bb^{-1}b_1$ for any $b_1 \in B$. Thus $b_1 \in bA$. Hence $B \subset bA$. The mapping $f(a) = ba$ (a in A, b is a fixed element of B) is a one-to-one and anti-order isomorphism by (ii) of Lemma 2, i.e. (i) holds. Since A is a subgroup of G, G is the sum $A \vee bA \vee dA \vee \cdots\cdots$ (disjoints), where $b \in B$, $d \in D$. Therefore D is the sum of some dA's. (iii) is obvious, by Lemma 1, 2.

A destroyer d is called *proper* if for any $a < b$ in G, $da \# db$.

[THEREM 2] *Let G be a po.c.g.II in which any destroyer is proper. Then A and B are convex subsets of G. (a subset S of G is convex if $\alpha, \beta \in S$ and $\alpha < x < \beta$ implies $x \in S$).*

[PROOF] Suppose $\alpha<x<\beta$ (α, β in A).

(i) If $x\epsilon B$, $\alpha x<\alpha\beta$. Since $\alpha<\beta$ implies $\alpha x>\beta x$, we have $\beta x<\alpha\beta$ i.e. $x<\alpha$. It is contrary to $\alpha<x$.

(ii) if $x\epsilon D$, $\alpha x<\alpha\beta$. Since $e<\alpha^{-1}x<\alpha^{-1}\beta$ and $\alpha^{-1}x\epsilon D$, we have $\alpha^{-1}x \# \alpha^{-1}x\alpha^{-1}\beta$. while $\alpha^{-1}x<\alpha^{-1}\beta<\alpha^{-1}x\alpha^{-1}\beta$, we have a contradiction. Hence $x\epsilon A$, i.e. A is a convex subset of G.

Suppose $\alpha<y<\beta$ (α, β in B). Since $\alpha^{-1}\epsilon B$, we have $e>y\alpha^{-1}>\beta\alpha^{-1}$. And e, $\beta\alpha^{-1}\epsilon A$, hence $y\alpha^{-1}\epsilon A$ i.e. $y\epsilon B$.

We proceed now to investigate the distributivity of the subset $A\vee B$.

[THEOREM 3] *Let G be a l.o.c.g.II in which $A<B$, and $e<x$ implies $x\epsilon A$. Then the subset $A\vee B$ is a distributive sublattice of G.*

[PROOF] Since, by Lemma 3, A is an l-subgroup, and A and B are anti-order isomorphic, we see: A and B are both distributive sublattices of G. If $x, y, a\epsilon A\vee B$, and $x\cap a=y\cap a$, $x\cup a=y\cup a$, then we can easily see $x=y$ for any case. Hence $A\vee B$ is a distributive sublattice of G.

§3 **Decomposition of po.c.g.II into s.o.c.g.II's** Throughout this section, we assume $B<A$. By a *simply ordered commutative group* (=s.o.c.g.) we mean a group G satisfying (a) G is simply ordered, and (b) $a\leqq b$ (a, b in G) implies $ac\leqq bc$ for any $c\epsilon G$, [3]. By a *simply ordered commutative group of the second kind* (=s.o.c.g.II), we mean a gronp G satisfying (α) G is simple ordered, and (β) every element of G is either a conserver or an inverter.

We can easily see that the set A of all conservers of a s.o.c.g.II becomes a s.o.c.g. Let B denote the set of all inverters of a s.o.c.g.II. And assume also $B<A$ in this section.

Now we shall investigate the condition that po.c.g.II is to be a cardinal product of some s.o.c.g.II's.

Before beginning our study, we state the following Lemma similarly to the way used by A.H. Clifford in [1].

we shall call an element ε *unit* element in G if $\varepsilon^2=e$.

[LEMMA 4] *For given a s.o.c.g.G_1, we can construct a s.o.c.g.II with an inverter unit element.*

[PROOF] Let ρ be a convex congruence relation (see [1]) in G_1 and k an element of G_1, such that $x\rho y$ implies $kx=ky$. Since G_1 is a group, we see $x=y$ if $x\rho y$. Let G_2 be the set of congruence classes of G_1 mod ρ, and let ϕ be the canonical mapping (see [1]) of G_1 onto G_2. The order relation in G_2 is defined as the followings: $\phi(x)<\phi(y)$ if and only if $x>y$ in G_1.

Let $G=G_1\vee G_2$ (disjoint) and order G so that $G_2<G_1$ Define product in G as the followings: for $x, y \in G_1$, $x\phi(y)=\phi(x)y=\phi(xy)$, $\phi(x)\phi(y)=xy$.

We now show that the above-defined G is a s.o.c.g.II as desired. To see that G is a group. We first investigete the associativity of G with respect to above product; for example

$$\phi(x)y\cdot\phi(z)=\phi(xy)\cdot\phi(z)=xy\cdot z=x\cdot yz=\phi(x)\cdot\phi(yz)=\phi(x)\cdot y\phi(x)$$
$$\phi(x)\phi(y)\cdot\phi(z)=xy\cdot\phi(z)=\phi(xy\cdot z)=\phi(x\cdot yz)=\phi(x)\cdot yz=\phi(x)\cdot\phi(y)\phi(z) \quad \text{etc.}$$

And for any $\phi(x) \in G_2$, we have $\phi(x)\phi(x^{-1})=e$ i.e. $\phi(x^{-1})=(\phi(x))^{-1}$. Hence we see that G becomes a group with respect to the product of G_1 and above-defined products. And we easily see that every element of G_1 is a conserver and every element of G_2 an inverter in G. And clearly, $\phi(e)$ is an inverter unit of G.

In the n-dim. Euclidean space, the subset

$$F_n=\{(1,1,\cdots,1),(-1,1,\cdots,1),(1,-1,\cdots 1),\cdots,(-1,-1,\cdots,-1)\}$$

becomes a l.o.c.g.II of order 2^n, if we define product and order relations in F_n as followings;

$$(a_1,\cdots\cdots, a_n)\cdot(b_1\cdots\cdots, b_n)=(a_1b_1,\cdots\cdots, a_nb_n)$$

$(a_1,\cdots\cdots, a_n)\leqq(b_1,\cdots\cdots, b_n)$ *if and only if* $a_i\leqq b_i$ *for all* i.

Clearly, the element $(-1,-1,\cdots\cdots,-1)$ is an inverter unit element of F_n. We call F_n *fundamental* l.o.c.g.II.

[THEOREM 4] *Let G be a po.c.g.II with an inverter unit element. And let G be group-isomorphic to a cardinal product of n s.o.c.g.II's with an inverte unit element. Then*

(i) *The set A of all conservers of G is group-isomorphic to a cardinal product of n s.o.c.g.'s, and $(G:A)=2^n$.*

(ii) *There exists a subgroup $\mathcal{F}_n=\{f_1(=e),f_2,\cdots\cdots,f_{2^n}\}$ of G such that $\mathcal{F}_n$ is isomorphic to F_n, and $f_if_j \notin A$ $(i\neq j)$.*

Conversely, if (i) and (ii) hold in G, then we can construct a l.o.c.g.II which is a cardinal product of n s.o.c.g.II's such that is group-isomorphic to G.

[PROOF] Let G be a cardinal product $(=\Pi G_i)$ of n s.o.c.g.II G_i's. Let A_i be the set of all conservers of G_i. Then we easily see that A is cardinal product ΠA_i of A_i's. Since A_i is a s.o.c.g., A is the cardinal product of n s.o.c.g.'s. Let ε_i, e_i be an inverter unit, an identity of G_i, respectively. Then the subset $\mathcal{F}_n=\{(a_1, \cdots\cdots, a_n) \mid a_i=e_i \text{ or } \varepsilon_i\}$ of ΠG_i is a sub-l.o.c.g.II of order 2^n, and moreover $\mathcal{F}_n$ is isomorphic to F_n. If $(a_1, \cdots\cdots, a_n)$ and $(b_1, \cdots\cdots, b_n)$ are two distinct elements of $\mathcal{F}_n$, then $(a_1, \cdots\cdots, a_n)\cdot(b_1, \cdots\cdots, b_n) \notin \Pi A_i$. Since $(a_1, \cdots\cdots, a_n)^{-1} =(a_1, \cdots\cdots, a_n)$ for any $(a_1, \cdots\cdots, a_n) \in \mathcal{F}_n$, $(a_1, \cdots\cdots, a_n)\not\equiv(b_1, \cdots\cdots, b_n)$ for mod ΠA_i. On the other hand, if $(x_1, \cdots\cdots, x_n) \in \Pi G_i$, then there exists some elements $(a_1, \cdots\cdots. a_n)$ of $\mathcal{F}_n$ such that $(x_1, \cdots\cdots, x_n)\equiv(a_1, \cdots\cdots, a_n)$ for mod A. Therefore $(G:A)=2^n$. Hence (i) and (ii) hold in G.

Conversely, Assume (i) and (ii) hold in given po.c.g.II G. By (ii), the set A of G is group-isomorphic to a cardinal product ΠA_i of n s.o.c.g. A_i's. By Lemma 5, we can construct a s.o.c.g.II G_i with an inverter unit element from each A_i. Now we must prove that G is group-isomorphic to the cardinal product ΠG_i *of* G_i's. To see this, by the foregoing way, we make $\overline{\mathcal{F}}_n$ of ΠG_i, so that $\overline{\mathcal{F}}_n \approx F_n$. By (ii), $\mathcal{F}_n$ of G is isomorphic to F_n. Thus $\overline{\mathcal{F}}_n \approx \mathcal{F}_n$. Since $G=A \vee f_2A \vee \cdots\cdots \vee f_{2^n}A$, where $f_i \in \mathcal{F}_n$, and $\Pi G_i=(\Pi A_i) \vee \boldsymbol{f}_2(\Pi A_i) \vee \cdots\cdots \vee \boldsymbol{f}_{2^n}(\Pi A_i)$, where $\boldsymbol{f}_i \in \overline{\mathcal{F}}_n$, the mapping: $f_i a \to \boldsymbol{f}_i \boldsymbol{a}$ is a group-isomorphism of G onto ΠG_i (a in A, $\boldsymbol{a}$ in ΠA_i). Where f_i corresponds to $\boldsymbol{f}_i$ by $\mathcal{F}_n \approx \overline{\mathcal{F}}_n$, and a corresponds to $\boldsymbol{a}$ by $A \approx \Pi A_i$. For, if $f_j b \to \boldsymbol{f}_j \boldsymbol{b}$ (b in A, $\boldsymbol{b}$ in ΠA_i), then $(f_i a)\cdot(f_j b)=f_k ab= f_k c \to \boldsymbol{f}_k \boldsymbol{c}=\boldsymbol{f}_i \boldsymbol{f}_j \boldsymbol{a}\boldsymbol{b}=(\boldsymbol{f}_i \boldsymbol{a})\cdot(\boldsymbol{f}_j \boldsymbol{b})$, where $f_i f_j=f_k$ in $\mathcal{F}_n$, $ab=c$ in A. Clearly, the mapping is one-to-one. Hence G is group-isomorphic to ΠG_i.

§ 4 Examples

[EXAMPLE 1] Let $E_n=\{(a_1, \cdots\cdots, a_n) \mid a_i(\neq 0) \text{ is a real number}\}$. And we define order and porduct relations in E_n as the followings;

$$(a_1, \cdots\cdots, a_n)\cdot(b_1, \cdots\cdots, b_n) = (a_1b_1, \cdots\cdots, a_n b_n)$$

$$(a_1, \cdots\cdots, a_n) \leqq (b_1, \cdots\cdots, b_n) \text{ if and only } if\ a_i \leqq b_i \text{ for all } i.$$

Then E_n is a l. o. c. g. II with destroyers which are not proper.

[EXAMPLE 2] In the E_n, if we define order relation as followings $(a_1, \cdots\cdots, a_n) \leqq (b_1, \cdots\cdots, b_n)$ if and only if either $a_i = b_i$ for all i or $a_i < b_i$ (*but* $a_i \neq b_i$) for all i. Then E_n is a l. o. c. g. II. And every destroyer is proper.

[EXAMPLE 3] Let G be the set of all one valued real functions $f(x)$ and it's inverse function $f^{-1}(x)$ defined on $[0,1]$ which are having at most finite number discontinuous points, and $f(x) \neq 0$ and $f^{-1}(x) \neq 0$ for all $x \in [0,1]$. Then G becomes a group under ordinary product of functions. Moreover one defines the order in G such that $f(x) \leqq g(x)$ means $f(x) \leqq g(x)$ for all x on $[0,1]$. Then G is a l. o. c. g. II with destroyers which are not proper.

[EXAMPLE 4] In the G of example 3, one defines the order in G such that $f(x) \leqq g(x)$ means either $f(x) = g(x)$ or $f(x) < g(x)$ for all x in $[0,1]$. Then every destroyer in G is proper.

Dec. 21, 1960

Mathematical Department
Liberal Arts and Science College
Kyungpook University.

REFERENCES

[1] A.H. Cifford, *Ordered commutative semigroups of the second kind*, Proc. Amer. Math. Soc. vol. 9 (1958) pp. 682-687

[2] A.H. Cifford, *Totally ordered commutative semigroups*, Bull. of Amer. Math. Soc. vol. 64 (1959).

[3] G. Birkhoff, *Lattice thory*, rev. ed. Now York, (1948)