THE CORRESPONDENCES OF THE FUNDAMENTAL FRAMES ON THE PARAMETER GROUP MANIFOLDS

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0. Introduction.

We consider the first parameter group manifold $\mathcal{G}^{(+)}$ and the second parameter group manifold $\mathcal{G}^{(-)}$, given $(+)$-connection $L^{\alpha}_{\beta\gamma}$ and $(-)$-connection $\overline{L}^{\alpha}_{\beta\gamma}$ by N. Horie respectively[2]. The purpose of this paper is to study on some properties of the correspondence of the fundamental frames in the sense of the operation, so called it the extended exterior differentiation by H. Flanders [1].

Let us take the followings as the equations of the combinations in the continuous transformation group of dimension r:

$$a^{\alpha}_{3}=\varphi^{\alpha}(a_{1},a_{2}),\quad (\alpha=1,\cdots\cdots,r).$$

Then we may consider the epuations

$$\frac{\partial a^{\alpha}_{3}}{\partial a^{\beta}_{2}}=A^{\alpha}_{b}(a_{3})A^{b}_{\beta}(a_{2}),\quad \frac{\partial a^{\alpha}_{3}}{\partial a^{\beta}_{1}}=\overline{A}^{\alpha}_{b}(a_{3})\overline{A}^{b}_{\beta}(a_{1})$$

as the fundamental equations of the parameter groups $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$ respectively, where determinant $|A^{\alpha}_{b}|$ is not zero and $||A^{b}_{\alpha}||$ is the inverise matrx of $||A^{\alpha}_{b}||$, and $\overline{A}$'s are similar to A 's.

Let the *fundamental frames* A_{α}, $\overline{A}_{\alpha}$ $(\alpha=1,\cdots\cdots,r)$ is $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$ are defined by the pairs $(A^{1}_{\alpha},\ldots\ldots,A^{r}_{\alpha})$ and $(\overline{A}^{1}_{\alpha},\ldots\ldots,\overline{A}^{r}_{\alpha})$ in the same local coordinate neighborhood U respectively, then we may consider the *dual bases* D^{α} and $\overline{D}^{\alpha}$ $(\alpha=1,\cdots\cdots,r)$ and define the extended exterior differentiation as H. Flanders [1], [3].

Using the matrix notations, we shall set

$$A=\begin{pmatrix}A_{1}\\ \vdots\\ A_{r}\end{pmatrix},\qquad D=(D^{1},\cdots\cdots,D^{r}).$$

Denoting the *fundamental displacement, the connection form, the torsion form* and the *curvature form* by dP, Ω, T and Θ on $\mathcal{G}^{(+)}$ respectively, then we may refer to be defined by the following equations [3]:

(0. 1) $\qquad dP=DA,$

(0. 2) $\qquad dA=\Omega A,\ \Omega=||w^{\alpha}_{\beta}||,\qquad$ (w^{α}_{β} are 1-forms),

(0. 3) $\qquad d^{2}P=d(dP)=TA,\ T=dD-D\Omega,\quad$ (T are 2-forms),

(0. 4) $dT=D\Theta-T\Omega,\ \Theta=d\Omega-\Omega^2$ (Θ are 2-forms).

And on $\mathcal{G}^{(-)}$, they are defined by the same manner.

Now, we can awake that the both curvature forms on $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$ vanish, since the curvature tensor fields $L^{\alpha}_{\beta\gamma\delta}$ and $\overline{L}^{\alpha}_{\beta\gamma\delta}$ vanish, and hence, on $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$, Θ and $\overline{\Theta}$ are zero matrices, and thus, (0. 4) are reduced to (0. 5)

(0. 5) $dT=-T\Omega,\quad d\Omega=\Omega^2.$

1. The linear correspondences of the fudamental frames between $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$

Let A and $\overline{A}$ be two fundamental frames of $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$ defined in the same local coordinate neighborhood U, and we suppose that there exists a linear correspondence

(1. 1) $\overline{A}=sA$

between the parameter group manifolds $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$, where s is a nonsingular matrix everywhere on U, whose elements are s^{α}_{β} $(\alpha,\beta=1,\cdots\cdots,r)$. Then the bases D and $\overline{D}$ of the given frames are related by

(1. 2) $\overline{D}=Ds^{-1}$,

being a consequence of $dP=DA=\overline{D}\,\overline{A}$. And from $d^2P=TA=\overline{T}\overline{A}$, the torsion forms T and $\overline{T}$ are also related by

(1. 3) $\overline{T}=Ts^{-1}$.

If we differentiate (1. 1), we have

$$d\overline{A}=\overline{\Omega}\,\overline{A}=dsA+s\Omega A,$$

and hence,

(1. 4) $\overline{\Omega}s=ds+s\Omega$ or $\overline{\Omega}=dss^{-1}+s\Omega s^{-1}$.

Writting this by the element form, it is

(1. 5) $\overline{w}^{\alpha}_{\beta}=ds^{\alpha}_{\tau}{}'s^{\tau}_{\beta}+s^{\alpha}_{\tau}w^{\tau}_{\sigma}{}'s^{\sigma}_{\beta}$

where $'s$'s are defined by

$$'s^{\alpha}_{\beta}=(cofactor\ of\ s^{\beta}_{\alpha}\ in\ s)/|s|,$$

and consequently,

(1. 6) $s^{\alpha}_{\tau}\ 's^{\tau}_{\beta}=\delta^{\alpha}_{\beta}$,

On account of $w^{\alpha}_{\beta}=L^{\alpha}_{\beta\tau}D^{\tau}$ and $\overline{L}^{\alpha}_{\beta\gamma}=L^{\alpha}_{\gamma\beta}$ on $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$[2], using of (1. 2) and contracting s^{β}_{σ} with respect to β, we get

$$\frac{\partial s^{\alpha}_{\beta}}{\partial a^{\gamma}}=L^{\alpha}_{\sigma\tau}{}'s^{\sigma}_{\gamma}\,s^{\tau}_{\beta}-L^{\tau}_{\beta\gamma}s^{\alpha}_{\tau}. \tag{1. 7}$$

Differentiating (1. 6) by a^{γ} and using of (1. 7), we have

$$\frac{\partial' s^{\alpha}_{\beta}}{\partial a^{\gamma}}=-L^{\tau}_{\sigma\beta}{}'s^{\sigma}_{\gamma}\;s^{\alpha}_{\tau}+L^{\alpha}_{\tau\gamma}{}'s^{\tau}_{\beta}. \tag{1. 8}$$

If we differentiate (1. 7) and (1, 8) by a^{δ}, interchange γ and δ, and subtract each other, since the curvature tensor of $\mathcal{G}^{(+)}$ vanishes, we obtain the followings as the conditions of the complete integrability of (1. 7) and (1. 8) respectively:

$$\left[\frac{\partial L^{\alpha}_{\sigma\tau}}{\partial a^{[\gamma}}{}'s^{\sigma}_{\delta]}+L^{\alpha}_{[\lambda|\sigma|}L^{\sigma}_{\mu]\tau}{}'s^{\lambda}_{\gamma}{}'s^{\mu}_{\sigma}-(L^{\alpha}_{\sigma\tau}s^{\sigma}_{\lambda}+L^{\alpha}_{\lambda\tau})\;L^{\lambda}_{\mu[\gamma}{}'s^{\mu}_{\delta]}\right]s^{\tau}_{\beta}+2L^{\tau}_{\sigma[\gamma}L_{\sigma|\beta|\delta]}s^{\alpha}_{\tau}=0^{(*)}, \tag{1. 9}$$

$$\left[\frac{\partial L^{\tau}_{\sigma\beta}}{\partial a^{[\gamma}}{}'s^{\sigma}_{\delta]}+(L^{\tau}_{\lambda\beta}+L^{\tau}_{\sigma\beta}{}'s^{\sigma}_{\lambda})L^{\lambda}_{\mu[\gamma}{}'s^{\mu}_{\delta]}-L^{\lambda}_{\sigma\beta}\,L^{\tau}_{[\lambda\mu]}{}'s^{\sigma}_{\gamma}{}'s^{\mu}_{\delta}\right]'s^{\alpha}_{\tau}+2L^{\alpha}_{\lambda[\gamma}L^{\lambda}_{|\tau|\delta]}{}'s^{\beta}_{\tau}=0. \tag{1.10}$$

Thus, we may assert that, for the linear correspondence from $\mathcal{G}^{(+)}$ to $\mathcal{G}^{(-)}$ and its inverse to exist, it is necessary and sufficient that (1. 9) and (1.10) hold respectively.

On the other hand, it is well known that the existence of the linear correspondence (1. 1) from $\mathcal{G}^{(+)}$ to $\mathcal{G}^{(-)}$ is equivalent to the one of the solutions of the (1. 4), and consequently, it must hold $d^2s=0$. If we differentiate (1. 4), we have

$$d^2s=d\overline{\Omega}s-\Omega ds-ds\Omega-sd\Omega. \tag{1.11}$$

In consequence of $\overline{L}^{\alpha}_{\beta\gamma\delta}=0$, it is shown by matrix form that

$$d\overline{\Omega}=\overline{\Omega}^2, \tag{1.12}$$

and hence, (1. 11) is reduced to

$$d^2s=\overline{\Omega}^2s-\overline{\Omega}\;(\overline{\Omega}s-s\Omega)-(\overline{\Omega}s-s\Omega)\Omega-s\Omega^2=0.$$

Thus, it obtains that there exist the solutions of (1. 4) always, and consequently, (1. 9) holds identically.

Next, since $|s|\neq 0$, we have $A=s^{-1}\overline{A}$ from (1. 1), and differentiating it, we have

(*) *$K_{[\alpha\beta]}$ denotes $K_{\alpha\beta}-K_{\beta\alpha}$.*

$$\Omega A = ds^{-1}sA + s^{-1}\Omega sA$$

or

(1. 13) $ds^{-1} = \Omega s^{-1} - s^{-1}\Omega.$

And differentinting (1. 13) too, we have from (1. 12) and (1. 3) $d^2 s^{-1}=0$, and hence, there exist the solutions of (1. 13), and consequently, we obtain that (1. 10) holds identically. From the above result, we obtain the following:

THEOREM 1 *There exists always the linear correspondence between the frames A and $\bar{A}$ from $\mathscr{G}^{(+)}$ to $\mathscr{G}^{(-)}$ and its inverse,* (1.9) *and* (1.10) *are satisfied identically.*

If we differentiate (1. 3), using of (1. 3), then we have

(1.14) $d\overline{T} = -\overline{T}\overline{\Omega}$

and

(1.15) $d^2\overline{T}=0$

And from (1.12), we have

(1.16) $d\overline{\Omega}^{2k}=0,\ d\overline{\Omega}^{2k+1}=\overline{\Omega}^{2k+2},\ (k\geqq 0).$

From (1.12), (1.14), (1.15), (1.16) and the results of the previous paper [3], we have the following:

THEOREM 2. *For the linear correspondence between the frames A and $\bar{A}$ from $\mathscr{G}^{(+)}$ to $\mathscr{G}^{(-)}$, the identities* (1.12), (1.14), (1.15) *and* (1.16) *are preserved.*

2. The linear correspondence of the fundamental frames on $\mathscr{G}^{(0)}$.

We are well known that the parameter group manifold $\mathscr{G}^{(0)}$ is defined by the connection [2]

(2. 1) $w^\alpha_\beta = \Gamma^\alpha_{\beta\gamma} D^\gamma$

where

$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{2}(L^\alpha_{\beta\gamma} + \bar{L}^\alpha_{\beta\gamma}),\quad \bar{L}^\alpha_{\beta\gamma} = L^\alpha_{\gamma\beta}.$$

Then, on $\mathscr{G}^{(0)}$, it may assert that

(2. 2) $\Gamma^\alpha_{\beta\gamma} = \Gamma^\alpha_{\gamma\beta},$

and the frames on $\mathscr{G}^{(0)}$ are A or $\bar{A}$. In this section, we shall investigate the linear correspondence between two frames.

Let us suppose that their frames A and $\overline{A}$ are the related by (1. 1), and then, their dual bases D and $\overline{D}$ are (1. 2). From (1. 1), we have (1. 4). Since $\overline{w}^\alpha_\beta = \Gamma^\alpha_{\beta\gamma}\, {}^{\prime}s^\gamma_\tau\, D^\tau$ from (2. 1) and (2. 2), writting (1. 4) by element form, we have

$$ds^\alpha_\beta = \overline{w}^\alpha_\sigma s^\sigma_\beta - s^\alpha_\sigma w^\sigma_\beta,$$

and thus,

$$(2.\ 3) \qquad \frac{\partial s^\alpha_\beta}{\partial a^\gamma} = \Gamma^\alpha_{\sigma\tau} s^\sigma_\beta\, {}^{\prime}s^\tau_\gamma - \Gamma^\sigma_{\beta\gamma} s^\alpha_\sigma.$$

Differentiating (1. 6) by a^γ, we obtain

$$(2.\ 4) \qquad \frac{\partial\, {}^{\prime}s^\alpha_\beta}{\partial a^\gamma} = -\Gamma^\sigma_{\beta\tau} s^\tau_\gamma\, {}^{\prime}s^\alpha_\sigma + \Gamma^\alpha_{\tau\gamma}\, {}^{\prime}s^\tau_\beta.$$

If we differentiate (2. 3) and (2. 4) by a^δ interchange γ and δ, and subtract each other, we obtain the followings as the conditions of the complete integrability of (2. 3) and (2. 4) respectively:

$$(2.\ 5) \qquad \left(\frac{\partial \Gamma^\alpha_{\sigma\tau}}{\partial a^{[\gamma}} + \Gamma^\alpha_{\sigma\lambda}\Gamma^\lambda_{\tau[\gamma}\right){}^{\prime}s^\tau_{\delta]} s^\sigma_\beta + \Gamma^\lambda_{\mu[\sigma}\Gamma^\alpha_{\tau]\lambda} s^\mu_\beta\, {}^{\prime}s^\sigma_\gamma\, {}^{\prime}s^\tau_\delta$$
$$+ \Gamma^\alpha_{\sigma\tau}\Gamma^\lambda_{\mu[\gamma} s^\mu_{\delta]} s^\sigma_\beta\, {}^{\prime}s^\tau_\lambda - 2\Gamma^\lambda_{\beta[\gamma}\Gamma^\sigma_{\delta]\lambda} s^\alpha_\sigma + \Gamma^\sigma_{\beta\gamma\delta} s^\alpha_\sigma = 0.$$

$$(2.\ 6) \qquad \left(\frac{\partial \Gamma^\sigma_{\beta\tau}}{\partial a^{[\gamma}} s^\tau_{\delta]} + \Gamma^\sigma_{\beta\tau}\Gamma^\tau_{\lambda\mu}\, {}^{\prime}s^\mu_{[\gamma} s^\lambda_{\delta]} + \Gamma^\tau_{\beta[\lambda}\Gamma^\sigma_{\mu]\tau} s^\lambda_\gamma\, s^\mu_\delta\right){}^{\prime}s^\alpha_\sigma$$
$$-2\Gamma^\lambda_{\tau[\gamma}\Gamma^\alpha_{\delta]\lambda}\, {}^{\prime}s^\tau_\beta - \Gamma^\alpha_{\tau\gamma\delta}\, {}^{\prime}s^\tau_\beta = 0.$$

Thus we may assert that, for the linear correspondence from $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$ and its inverse to exist, it is necessary and sufficient that (2. 5) and (2. 6) hold respectively.

On the other hand, if we differentiate (1. 4), we have by using of (0. 4) and (1, 4), (1,13), since they are satisfied on $\mathcal{G}^{(0)}$,

$$d\overline{\Omega} = \overline{\Omega}^2 + s\Theta s^{-1} + d^2 s s^{-1}.$$

In consequence of $\overline{\Theta} = s\Theta s^{-1}$ [1], this is reduced to

$$d\overline{\Omega} = \overline{\Omega}^2 + \overline{\Theta} + d^2 s s^{-1}.$$

Then, we obtain that $d^2 s = 0$ is equivalent to

$$(2,\ 7) \qquad d\overline{\Omega} = \overline{\Omega}^2 + \overline{\Theta}.$$

Differentiating $ss^{-1} = E$ repeatively twice, E being identity matrix, then we have

$$d^2 s s^{-1} + s d^2 s^{-1} = 0,$$

and consequently, that $d^2s=0$ is equivalent to $d^2s^{-1}=0$. Hence we have the followings

THOEREM 3. *There exists the linear correspondence between the frames A and $\overline{A}$ on $\mathcal{G}^{(0)}$ and its inverse, if, and only if, the curvature form* (2.7) *is preserved. Therefore* (2. 5) *and* (2. 6) *are equivalent to* (2.7).

Next, since $T=0$ in $\mathcal{G}^{(0)}$, we have

$$dD=D\Omega. \tag{2, 8}$$

From this, we obtain the followings:

$$d^{2m}D=(-1)^m D\Theta^m, \qquad d^{2m+1} D=(-1)^m D\Theta^m\Omega. \qquad (m\geqq 1) \tag{2, 6}$$

In fact, for $m=1$, it is satisfied by differentiation of (2, 8) to use of (0, 4), for the general case it can be shown by induction on using of the Bianchi identities [3]. And we may compute

$$d\overline{\Theta}=d(s\Theta s^{-1})=\overline{\Omega}s\Theta s^{-1}-s\Theta s^{-1}\overline{\Omega}$$
$$=\overline{\Omega}\,\overline{\Theta}-\overline{\Theta}\,\overline{\Omega},$$

and generally,

$$d\overline{\Theta}^m=\overline{\Omega}\,\overline{\Theta}^m-\overline{\Theta}^m\,\overline{\Omega}. \qquad (m\geqq 1). \tag{2.10}$$

Differentiating (1. 2), we have

$$dD=dDs^{-1}-D(\Omega s^{-1}-s^{-1}\overline{\Omega})=\overline{D}\overline{\Omega},$$

and using of (2. 10), then we have that (2. 9) is preserved by (1. 1). Hence we obtain the following:

THEOREM 4. *For the linear correspondence between the frames A and $\overline{A}$ on $\mathcal{G}^{(0)}$, the Bianchi identities* (2.10), (2. 7), (2. 8) *and* (2. 9) *are preserved.*

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