PSEUDO UNITARY NORMALS AND GAUSSIAN FORMULAE FOR THE SUB-KAEHLERIAN MANIFOLD

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1. Introduction.

We consider a Kaehlerian manifold K_m whose complex analytic structure $(u^\alpha, \overset{*}{u}{}^\alpha)$ $(\overset{*}{u}{}^\alpha = u^{\alpha^*} = conj.$ of u^α; $\alpha,\beta,\gamma\cdots=1,2,\cdots,m$; $\alpha^*,\beta^*,\gamma^*,\cdots=1^*,2^*,\cdots,m^*$; $\alpha^*=m+\alpha)$ was given by the relations [1]

$$(1.\ 1)\qquad u^\alpha = y^\alpha + iy^{\alpha^*}, \qquad \overset{*}{u}{}^\alpha = y^\alpha - iy^{\alpha^*}$$

where $(y^A)=(y^\alpha, y^{\alpha^*})$ $(A,B,C,\cdots=1,2,\cdots,m,1^*,2^*,\cdots,m^*)$ is a system of coordinate neighborhoods of a real 2m-dimensional Riemannian manifold X_{2m}.

By considering a X_{2n}, of coordinate neighborhoods $x^K=(x^k, x^{\bar k})$ $(k,j,l,\cdots=1,2,\cdots,n$; $\bar k,\bar j,\cdots=\bar 1,\bar 2,\cdots,\bar n$; $K,J,L,\cdots=1,2,\cdots,n,\bar 1,\bar 2,\cdots,\bar n$; $\bar k=n+k)$, immersed in above X_{2m}, we can consider that the K_n whose complex analytic structure $(z^k, \bar z^k)$ $(\bar z^k = z^{\bar k} = conj.$ of $z^k)$ was given by the relations [1]

$$(1.1')\qquad z^k = x^k + ix^{\bar k}, \qquad \bar z^k = x^k - ix^{\bar k}$$

is also immersed in above K_m.

For the real Riemannian manifold X_{2m}, we denote the metric tensor and Christoffel symbols by the notations a_{AB}, ${}_{(a)}\{{}^{A}_{BC}\}$, respectively and for the above Kaehlerian manifold K_m, by h_{AB}, ${}_{(h)}\Gamma^A_{BC}$, moreover for the X_{2n}, by b_{JK}, $\{{}^{J}_{KL}\}$, for the K_n, by g_{JK}, Γ^J_{KL} respectively, then we have by (1.2) of [1]

$$(1.\ 2)\qquad h_{AB}(u)=\frac{\partial y^C}{\partial u^A}\frac{\partial y^D}{\partial u^B}a_{CD}(y), \qquad g_{JK}(z)=\frac{\partial x^H}{\partial z^J}\frac{\partial x^L}{\partial z^K}b_{HL}(x)$$

and we have the following relations

$$(1.\ 3)\qquad a_{\alpha\beta}=a_{\alpha^*\beta^*},\quad a_{\alpha\beta^*}=-a_{\alpha^*\beta},\quad b_{jk}=b_{\bar j\bar k},\quad b_{j\bar k}=-b_{\bar jk},$$

$$(1.\ 4)\qquad h_{\alpha\beta^*}=\frac{1}{2}(a_{\alpha\beta}+ia_{\alpha\beta^*}),\qquad h^{\alpha\beta^*}=2(a^{\alpha\beta}-ia^{\alpha\beta^*}),\quad (conj.)$$

$$g_{j\bar k}=\frac{1}{2}(b_{jk}+ib_{j\bar k}),\qquad g^{j\bar k}=2(b^{jk}-ib^{j\bar k}),\qquad (conj.)$$

$$(1.\ 5)\qquad {}_{(h)}\Gamma^\alpha_{\beta\gamma}={}_{(a)}\left\{{}^{\alpha}_{\beta\gamma}\right\}-i\,{}_{(a)}\left\{{}^{\alpha}_{\beta\gamma^*}\right\},\quad \Gamma^h_{jk}=\left\{{}^{h}_{jk}\right\}-i\left\{{}^{h}_{j\bar k}\right\}.\quad (conj.)$$

The main purpose of this paper is representation of the normals to K_n and the Gaussian formulae for the submanifold K_n of K_m by means of the relations [2]

$$(1.6)\qquad \frac{\partial}{\partial y^\alpha}=\frac{\partial}{\partial u^\alpha}+\frac{\partial}{\partial \overset{*}{u}{}^\alpha},\qquad \frac{\partial}{\partial y^{\alpha^*}}=i\Big(\frac{\partial}{\partial u^\alpha}-\frac{\partial}{\partial \overset{*}{u}{}^\alpha}\Big),$$

$$\frac{\partial}{\partial x^k}=\frac{\partial}{\partial z^k}+\frac{\partial}{\partial \bar z^k},\qquad \frac{\partial}{\partial x^{\bar k}}=i\Big(\frac{\partial}{\partial z^k}-\frac{\partial}{\partial \bar z^k}\Big).$$

If we assume that the K_n is a complex analytic submanifold of K_m i.e.

$$u^\alpha=u^\alpha(z^1,\ z^2,\cdots,z^n),\quad \overset{*}{u}{}^\alpha=\overset{*}{u}{}^\alpha(\bar z^1,\ \bar z^2,\cdots,\bar z^n),$$

then by the relation (1.1) and (1.6), we have easily

$$(1.7)\qquad \frac{\partial y^\alpha}{\partial x^j}=\frac{\partial y^{\alpha^*}}{\partial x^{\bar j}}=\frac{1}{2}\Big(\frac{\partial u^\alpha}{\partial z^j}+\frac{\partial \overset{*}{u}{}^\alpha}{\partial \bar z^j}\Big),$$

$$\frac{\partial y^\alpha}{\partial x^{\bar j}}=-\frac{\partial y^{\alpha^*}}{\partial x^j}=\frac{i}{2}\Big(\frac{\partial u^\alpha}{\partial z^j}-\frac{\partial \overset{*}{u}{}^\alpha}{\partial \bar z^j}\Big).$$

Conversely if the relations (1.7) are satisfied, the K_n is a complex analytic submanifold of K_m.

Therefore we have the following:

A necessary and sufficient condition that a Kaehlerian manifold K_n which was dirctly complexified from a real Riemannian manifold X_{2n}, of coordinates x^K, immersed in a X_{2m} of coordinates y^A, be a complex analytic submanifold of K_m which was directly complexified from X_{2m}, is the relations (1.7).

By means of the relations (1.2), (1.6) and (1.7) we can easily see

$$(1.8)\qquad g_{j\bar k}=h_{\alpha\beta^*}\frac{\partial u^\alpha}{\partial z^j}\frac{\partial \overset{*}{u}{}^\beta}{\partial \bar z^k}.$$

Throughout this paper we assume that above K_n is a cmplex analytic submanifold of K_m.

2. Pseudo unitary normals.

Let us denote the 2(m-n) mutually orthogonal unit normals to X_{2n}, by

$$N_{M|}{}^A=(N_{M|}{}^\alpha,\ N_{M|}{}^{\alpha^*}),\ (M,P,\cdots=1,\cdots,m-n,m-n+1,\cdots,2(m-n))$$

then

$$a_{AB}N_{M|}{}^A N_{P|}{}^B=\delta_{MP}.$$

By putting

$$(2.1)\qquad \xi_{M|}{}^\alpha=N_{M|}{}^\alpha+iN_{M|}{}^{\alpha^*},$$

$$\overset{*}{\xi}_{M|}{}^{\alpha}=N_{M|}{}^{\alpha}-iN_{M|}{}^{\alpha*} \qquad (\overset{*}{\xi}_{M|}{}^{\alpha}=\xi_{M|}{}^{\alpha*})$$

we have easily by the relations (1.3) and (1.4)

$$(2.\ 2) \qquad h_{\alpha\beta*}(\xi_{M|}{}^{\alpha}\overset{*}{\xi}_{P|}{}^{\beta}+\xi_{P|}{}^{\alpha}\overset{*}{\xi}_{M|}{}^{\beta})=\delta_{MP}$$

i.e.
$$\mathscr{R}(h_{\alpha\beta*}\xi_{M|}{}^{\alpha}\overset{*}{\xi}_{P|}{}^{\beta})=\frac{1}{2}\delta_{MP},$$

where $\mathscr{R}(Z)$ is the real part of Z, and if $M=P$ then,

$$(2.\ 2') \qquad h_{\alpha\beta*}\xi_{M|}{}^{\alpha}\overset{*}{\xi}_{M|}{}^{\beta}=\frac{1}{2}$$

therefore we define that $\xi_{M|}{}^{A}$ are ***pseudo unitary normals*** to K_n.

Moreover, by the relations

$$a_{AB}y^{A}{}_{,K}N_{M|}{}^{B}=0, \qquad \left(y^{A}{}_{,K}=\frac{\partial y^{A}}{\partial x^{K}}\right)$$

we have

$$(2.\ 3) \qquad h_{\alpha\beta*}(u^{\alpha}{}_{.k}\overset{*}{\xi}_{M|}{}^{\beta}+\overset{*}{u}{}^{\beta}{}_{.\bar{k}}\xi_{M|}{}^{\alpha})=0 \qquad \left(u^{\alpha}{}_{.k}=\frac{\partial u^{\alpha}}{\partial z^{k}} \quad (conj.)\right)$$

i.e.
$$\mathscr{R}((h_{\alpha\beta*}u^{\alpha}{}_{.k}\overset{*}{\xi}_{M|}{}^{\beta})=0.$$

By calculating directly from (1.6) and (2.1), we have

$$\frac{\partial\xi_{M|}{}^{\alpha}}{\partial\overset{*}{u}{}^{\gamma}}=\frac{1}{2}\left(\frac{\partial N_{M|}{}^{\alpha}}{\partial y^{\gamma}}-\frac{\partial N_{M|}{}^{\alpha*}}{\partial y^{\gamma*}}\right)+i\left(\frac{\partial N_{M|}{}^{\alpha}}{\partial y^{\gamma*}}+\frac{\partial N_{M|}{}^{\alpha*}}{\partial y^{\gamma}}\right),$$

$$\frac{\partial\overset{*}{\xi}_{M|}{}^{\alpha}}{\partial y^{\gamma}}=\frac{1}{2}\left(\frac{\partial N_{M|}{}^{\alpha}}{\partial y^{\gamma}}-\frac{\partial N_{M|}{}^{\alpha*}}{\partial y^{\gamma*}}\right)-i\left(\frac{\partial N_{M|}{}^{\alpha}}{\partial y^{\gamma*}}+\frac{\partial N_{M|}{}^{\alpha*}}{\partial y^{\gamma}}\right),$$

therefore we have the following:

If the pseudo unitary normals $\xi_{M|}{}^{A}$ are complex analytic, i.e.

$$\xi_{M|}{}^{\alpha}=\xi_{M|}{}^{\alpha}(u^{1},u^{2},\cdots,u^{m}), \qquad \overset{*}{\xi}_{M|}{}^{\alpha}=\overset{*}{\xi}_{M|}{}^{\alpha}(u^{1*},u^{2*},\cdots,u^{m*}),$$

then

$$(2.\ 4) \qquad \frac{\partial N_{M|}{}^{\alpha}}{\partial y^{\gamma}}=\frac{\partial N_{M|}{}^{\alpha*}}{\partial y^{\gamma*}}, \qquad \frac{\partial N_{M|}{}^{\alpha}}{\partial y^{\gamma*}}=-\frac{\partial N_{M|}{}^{\alpha*}}{\partial y^{\gamma}},$$

and vice versa.

For the complex analytic $\xi_{M|}{}^{A}$, by the assumption of §1, we have also

$$(2.\ 4') \qquad \frac{\partial N_{M|}{}^{\alpha}}{\partial x^{k}}=\frac{\partial N_{M|}{}^{\alpha*}}{\partial x^{\bar{k}}}, \qquad \frac{\partial N_{M|}{}^{\alpha}}{\partial x^{\bar{k}}}=-\frac{\partial N_{M|}{}^{\alpha*}}{\partial x^{k}}.$$

By denoting the tensor derivative with respect to y^{γ} and x^{k} by $;\gamma$ and $;k$ respectively and with respect to u^{γ} and z^{k} by $:\gamma$ and $:k$ respectively, then in virtue of (2. 4), (2. 5) and (1. 5) for the complex analytic pseudo unitary normal vectos $\xi_{M|}{}^{A}$, we have

$$(2.5)\qquad \begin{aligned} N_{M|}{}^{\alpha}{}_{;\gamma}=N_{M|}{}^{\alpha^*}{}_{;\gamma^*}=\frac{1}{2}(\xi_{M|}{}^{\alpha}{}_{:\gamma}+\overset{*}{\xi}_{M|}{}^{\alpha}{}_{:\gamma^*}),\\ N_{M|}{}^{\alpha}{}_{;\gamma^*}=-N_{M|}{}^{\alpha^*}{}_{;\gamma}=\frac{i}{2}(\xi_{M|}{}^{\alpha}{}_{:\gamma}-\overset{*}{\xi}_{M|}{}^{\alpha}{}_{:\gamma^*}),\end{aligned}$$

$$(2.6)\qquad \begin{aligned} N_{M|}{}^{\alpha}{}_{;k}=N_{M|}{}^{\alpha^*}{}_{;\bar{k}}=\frac{1}{2}(\xi_{M|}{}^{\alpha}{}_{:k}+\overset{*}{\xi}_{M|}{}^{\alpha}{}_{:\bar{k}}),\\ N_{M|}{}^{\alpha}{}_{;\bar{k}}=-N_{M|}{}^{\alpha^*}{}_{;k}=\frac{i}{2}(\xi_{M|}{}^{\alpha}{}_{:k}-\overset{*}{\xi}_{M|}{}^{\alpha}{}_{:\bar{k}}).\end{aligned}$$

Moreover, for the following vectors

$$(2.7)\qquad \vartheta_{MP|J}=a_{AB}N_{M|}{}^{A}\,y^{C}{}_{,J}\,N_{P|}{}^{B}{}_{;C}$$

by substituting (2.5), we have

$$(2.8)\qquad \begin{aligned} \vartheta_{MP|j}&=h_{\alpha\beta^*}(\xi_{M|}{}^{\alpha}\,\overset{*}{u}{}^{\gamma}{}_{,\bar{j}}\,\overset{*}{\xi}_{P|}{}^{\beta}{}_{:\gamma^*}+\overset{*}{\xi}_{M|}{}^{\beta}\,u^{\gamma}{}_{,j}\,\xi_{P|}{}^{\alpha}{}_{:\gamma})\\ &=2\mathscr{R}(h_{\alpha\beta^*}\,\xi_{M|}{}^{\alpha}\,\overset{*}{u}{}^{\gamma}{}_{,\bar{j}}\,\overset{*}{\xi}_{P|}{}^{\beta}{}_{:\gamma^*}),\\ \vartheta_{MP|\bar{j}}&=ih_{\alpha\beta^*}(-\xi_{M|}{}^{\alpha}\,\overset{*}{u}{}^{\gamma}{}_{,\bar{j}}\,\overset{*}{\xi}_{P|}{}^{\beta}{}_{:\gamma^*}+\overset{*}{\xi}_{M|}{}^{\beta}\,u^{\gamma}{}_{,j}\,\xi_{P|}{}^{\alpha}{}_{:\gamma})\\ &=-2i\mathscr{J}(h^{\alpha}{}_{\beta^*}\,\xi_{M|}{}^{\alpha}\,\overset{*}{u}{}^{\gamma}{}_{,\bar{j}}\,\overset{*}{\xi}_{P|}{}^{\beta}{}_{:\gamma^*}),\end{aligned}$$

where $\mathscr{J}(Z)$ is the imaginary part of Z.

Therefore, by putting

$$(2.9)\qquad \mu_{MP|j}=h_{\alpha\beta^*}\,\xi_{M|}{}^{\alpha}\overset{*}{u}{}^{\gamma}{}_{,\bar{j}}\overset{*}{\xi}_{P|}{}^{\beta}{}_{:\gamma^*},\qquad (conj)$$

we have the following relations

$$(2.10)\qquad \begin{aligned} \vartheta_{MP|j}=\mu_{MP|j}+\mu_{MP|\bar{j}},\qquad (\mu_{MP|\bar{j}}=conj.\ of\ \mu_{MP|j})\\ \vartheta_{MP|\bar{j}}=i(-\mu_{MP|j}+\mu_{MP|\bar{j}}),\end{aligned}$$

and

$$(2.11)\qquad \mu_{MP|j}=\frac{1}{2}(\vartheta_{MP|j}+i\vartheta_{MP|\bar{j}}).$$

3. Gaussian formulae.

The tensor derivatives of $y^{A}{}_{,k}$ are the following forms by the relations (1. 5) and (2. 4)

$$y^{\alpha}{}_{:jk}=-y^{\alpha}{}_{:\bar{j}\bar{k}}=\frac{1}{2}(u^{\alpha}{}_{:jk}+\overset{*}{u}{}^{\alpha}{}_{:\bar{j}\bar{k}}),$$

$$y^{\alpha}{}_{;j\bar{k}} = y^{\alpha}{}_{;\bar{j}k} = \frac{i}{2}(u^{\alpha}{}_{:jk} - \overset{*}{u}{}^{\alpha}{}_{:\bar{j}\bar{k}}),$$
(3. 1)
$$y^{\alpha^*}{}_{;jk} = -y^{\alpha}{}_{;j\bar{k}}, \quad y^{\alpha^*}{}_{;j\bar{k}} = y^{\alpha^*}{}_{;\bar{j}k} = y^{\alpha}{}_{;jk}, \quad y^{\alpha^*}{}_{;\bar{j}\bar{k}} = y^{\alpha}{}_{;j\bar{k}},$$

where

$$u^{\alpha}{}_{:jk} = \frac{\partial^2 u}{\partial z^j \partial z^k} + {}_{(h)}\Gamma^{\alpha}_{\beta\gamma}\, u^{\beta}{}_{,j}\, u^{\gamma}{}_{,k} - \Gamma^{l}_{jk}\, u^{\alpha}{}_{,l},$$

and from the equations

$$a_{AB}\, y^{A}{}_{;JK}\, y^{B}{}_{,L} = 0$$

we have easily

$$h_{\alpha\beta^*}(u^{\alpha}{}_{:jk}\, \overset{*}{u}{}^{\beta}{}_{,l} + \overset{*}{u}{}^{\alpha}{}_{:\bar{j}\bar{k}}\, u^{\beta}{}_{,l}) = 0, \tag{3. 2}$$

therefore we can regard that $u^{\alpha}{}_{:jk}$, of a vector in K_m, is pseudo normal to K_n.

The second fundamental tensor of X_{2n} immersed in X_{2m} is given by

$$\Omega_{M|JK} = y^{A}{}_{;JK}\, a_{AB}\, N_{M|}{}^{B} \tag{3. 3}$$

and by the relations (3.1), for the complex analytic vectors $\xi_{M|}{}^{A}$ we have

$$\Omega_{M|jk} = u^{\alpha}{}_{:jk}\, h_{\alpha\beta^*}\, \overset{*}{\xi}_{M|}{}^{\beta} + \overset{*}{u}{}^{\alpha}{}_{:\bar{j}\bar{k}}\, h_{\alpha^*\beta}\, \xi_{M|}{}^{\beta},$$
(3. 4)
$$\Omega_{M|j\bar{k}} = i(u^{\alpha}{}_{:jk}\, h_{\alpha\beta^*}\, \overset{*}{\xi}_{M|}{}^{\beta} - \overset{*}{u}{}^{\alpha}{}_{:\bar{j}\bar{k}}\, h_{\alpha^*\beta}\, \xi_{M|}{}^{\beta}),$$

and

$$\Omega_{M|\bar{j}k} = \Omega_{M|j\bar{k}}, \qquad \Omega_{M|\bar{j}\bar{k}} = -\Omega_{M|jk}. \tag{3.4'}$$

The mean curvare of X_{2n} in X_{2m} is given by

$$\mathrm{M} = \sum_{P} M_{P|}\mathrm{N}_{P|}$$

where

$$M_{P|} = \Omega_{P|JK}\, b^{JK}$$

and $\mathrm{N}_{P|}$ are the 2(m—n) mutually orthogonal unit normal vectors to X_{2n}, then by the relation (1.4) and (3.4′) we can easily see

$$M_{P|} = 0$$

therefore, we have the following:

When a Riemannian manifold X_{2n}, immersed in X_{2m}, is related to a complex analytic sub-Kaehlerian manifold K_n, immersed in K_{2m}, by (2.1), (2.2) and (1.5), *the mean curvature of X_{2n} vanishes.*

The Gaussian formulae for the submanifold X_{2n} of X_{2m} are gevin by

$$y^A{}_{;JK}=\sum_M \Omega_{M|JK} N_{M|}{}^A \tag{3. 5}$$

and by substituting (3.1) and (3.4) into (3.5), we have

$$\begin{aligned} u^\alpha{}_{:jk} &= \sum_M \Theta_{M|\bar{j}k}(\xi_{M|}{}^\alpha+\overset{*}{\xi}_{M|}{}^\alpha), \\ \overset{*}{u}{}^\alpha{}_{:\bar{j}\bar{k}} &= \sum_M \Theta_{M|j\bar{k}}(\xi_{M|}{}^\alpha+\overset{*}{\xi}_{M|}{}^\alpha), \end{aligned} \tag{3. 6}$$

where we have put

$$\begin{aligned} \Theta_{M|\bar{j}k} &= u^\alpha{}_{:jk}\, h_{\alpha\beta^*}\, \overset{*}{\xi}_{M|}{}^\beta, \\ \Theta_{M|j\bar{k}} &= \overset{*}{u}{}^\alpha{}_{:\bar{j}\bar{k}}\, h_{\alpha^*\beta}\, \xi_{M|}{}^\beta, \end{aligned} \tag{3. 7}$$

i.e. $\Theta_{M|j\bar{k}}=\frac{1}{2}(\Omega_{M|jk}+i\Omega_{M|j\bar{k}}),\qquad \Theta_{M|\bar{j}k}=\frac{1}{2}(\Omega_{M|jk}-i\Omega_{M|j\bar{k}}).$

The set of equations (3.6) *and* (3.7) *are the Gaussian formulae and the second fundamental tensor respectively for the sub-Kaehlerian manifold* K_n *of* K_m.

The tensor derivatives of $N_{M|}{}^A$ are given by

$$N_{M|}{}^A{}_{;J}=-\Omega_{M|JK}\, b^{KL}\, y^A{}_{,L}+\sum_P \vartheta_{PM|J} N_{P|}{}^A \tag{3. 9}$$

by substituting (2.6), (2.10) and (3.8) into (3.9), we have

$$\begin{aligned} \xi_{M|}{}^\alpha{}_{:k} &= -\Theta_{M|\bar{k}h}\, g^{h\bar{j}}\, \overset{*}{u}{}^\alpha{}_{.\bar{j}} + \sum_P \mu_{PM|\bar{k}}(\xi_{P|}{}^\alpha+\overset{*}{\xi}_{P|}{}^\alpha), \\ \overset{*}{\xi}_{M|}{}^\alpha{}_{:\bar{k}} &= -\Theta_{M|k\bar{h}}\, g^{\bar{h}j}\, u^\alpha{}_{.j}+\sum_P \mu_{PM|k}(\xi_{P|}{}^\alpha+\overset{*}{\xi}_{P|}{}^\alpha), \end{aligned} \tag{3.10}$$

and *the set of equations* (3.10) *are the tensor derivatives of* $\xi_{M|}{}^A$.

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