NOTES ON THE SET OF THE PARTIAL ORDERINGS

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Introduction.

It is well known that the set of the partitions of a set is a complete lattice. Here I will concern that the set of some partial orderings on a set forms an atomic complete Boolean lattice, in the section one. In the section two there are concerned with the relation between the set of the partitions of a set S and the set of the partial orderings on S.

In his paper [2], Ladislas Fuchs asserted that any normal partly ordered commutative group has an extension of this partly ordered commutative group and it has also the normality. So I am concerned with the set of the normal partly ordered commutative groups on a fixed group A, in the section three.

1. The set of the partly ordered sets on a fixed set S.

Let S be a fixed abstract set in the section one and two. P_α is a partly ordered set means P_α has a partial ordering $\leqq$ on S:

$x\leqq x$ in P_α for any $x\in S$; $x\leqq y$ in P_α and $y\leqq x$ in P_α implies $x=y$;

$x\leqq y$ in P_α and $y\leqq z$ in P_α implies $x\leqq z$ in P_α.

We say that P_α, P_β have *compatible partial orderings* if and only if for any $x\neq y$, $x\leqq y$ in P_α then $x\not\geqq y$ in P_β and $x\leqq y$ in P_β then $x\not\geqq y$ in P_α.

Let P be the set of all partly ordered sets on S. We give a relation $\leqq$ between elements P_α, P_β of P by $P_\alpha \leqq P_\beta$ in P if and only if $x\leqq y$ in P_α implies $x\leqq y$ in P_β. Then it is easily shown that the relation in P is a partial ordering, i.e. reflexive, antisymmetric and transitive relation.

P_α is said to be a *maximal* element in P if there exists no element P_β of P such that $P_\beta > P_\alpha$ in P. E. Szpilrajn has proved that every partial ordering defined on a set has a linear extension, [3]. Let L_α be a linear ordering on S then L_α is a maxial element in P.

P_α is said to be a minimal element in P if there exists no element P_β of P such that $P_\beta < P_\alpha$.

Let P_o be a partly ordered set on S such that for any 2 elements x, y of S $x \# y$ in P_o,[4]. Then P_o is a minimal element in P. Moreover P_o is the least element of P. Let us denote the least element in P by O, i.e. O is a partly ordered set on S such that for any x,y of O $x \# y$ in O.

Let P_α be a partly ordered set on S such that the fixed elements x, y of S

$x \leqq y$ in P_α and for arbitrary elements u, v of S distinct from x, y $x \# u$, $x \# v$, $u \# v$ in P_α. Obviously this partly ordered set covers O in P. Hence this type of partly ordered set are called by atom in P, and written by $A(x, y)$.

We denote by $A(y, x)$ as a dual of the partial ordered set $A(x, y)$. $A(y, x)$ is also an atom in P. And $A(x, y) \# A(y, x)$ in P.

$A(x, y), A(y, x)$ are not compatible partly ordered sets.

For any P_α, P_β of P we define a relation in P_γ as follows:

$x \leqq y$ in P_γ if and only if $x \leqq y$ in P_α and $x \leqq y$ in P_β.

Then P_γ is a partly ordered set on S. And P_γ is the meet of P_α and P_β in P, obviously. We denote the meet of P_α and P_β by $P_\alpha \cup P_\beta$. So that P is an atomic meet-semilattice with zero element O.

Let M_α be a set of $P_\alpha \in P$ such that the all elements in M_α are compatible each other. Denote by $M_\alpha = \{P_\alpha \mid \alpha \in A\}$. Thus $M_\alpha \subset P$.

LEMMA 1. *M_α is an atomic complete lattice.*

PROOF. Obviously O is contained in M_α. If $P_\alpha \in M_\alpha$ ($P_\alpha \neq O$) has elements x, y: $x \leqq y$ in P_α then there exists an atom $A(x, y)$ in M_α such that $A(x, y) \leqq P_\alpha$ in M_α. Hence M_α is an atomic meet-semilattice. M_α has a unit element L_α, linear ordered set compatible with P_α. If L_α, L'_α are linear ordered sets in M_α. Then since L_α, L'_α are linear ordered sets for arbitrary 2 elements x, y $x \leqq y$ or $x \geqq y$ in L_α, and $x \leqq y$ or $x \geqq y$ in L'_α. If $x \leqq y$ in L_α and $x \leqq y$ in L'_α then $L_\alpha \leqq L'_\alpha$ and $L'_\alpha \leqq L_\alpha$. So that $L_\alpha = L'_\alpha$. If $x \leqq y$ in L_α and $x \geqq y$ in L'_α then it is a contradiction to the assumption of M_α. Hence M_α has a unit element L_α.

For arbitrary $A_1 \subseteq A$, $\{P \in A_1\} \subset M_\alpha$. $\cap\{P_\alpha \mid \alpha \in A_1\}$ exists and a partly ordered set on S by

$$x \leqq y \text{ in } \cap\{P_\alpha \mid \alpha \in A_1\} \iff x \leqq y \text{ in } P_\alpha \text{ for all } \alpha \in A_1.$$

Obviously $\cap\{P_\alpha \mid \alpha \in A_1\} \in M_\alpha$. Thus M_α is an atomic complete lattice.

P_α, P_β are compatible partly ordered sets on S : P_α, $P_\beta \in M_\alpha$. We are going to consider with a new relation $\leqq$ in P_δ on S as follows:

$$x \leqq y \text{ in } P_\delta \iff x \leqq y \text{ in } P_\alpha \text{ or } x \leqq y \text{ in } P_\beta.$$

Then $\leqq$ in P_δ is a reflexive, antisymmetric relation but not always transitive. For, it is easily seen reflexive. $\leqq$ in P_δ is an antisymmetric relation because P_α, P_β are compatible. We have an example such that this relation $\leqq$ in P_δ is not always transitive.

EXAMPLE. Let x, y, z be 3 distinct elements of S. Let $P_\alpha = A(x, y)$ and $P_\beta = A(y, z)$, in P_δ $x \leqq y$ by $x \leqq y$ in P_α, and $y \leqq z$ in P_δ by $y \leqq z$ in P_β. But $x \# z$ in P_δ because $x \# z$ in P_α and P_β. Thus the relation $\leqq$ in P_δ is not transitive.

We can now obtain the following 3 lemmas easily.

LEMMA 2. *Let P_δ have a transitive relation $\leqq$:*

$x \leqq y$ in P_δ if and only if $x \leqq y$ in P_α or $x \leqq y$ in P_β.
Then P_δ is a partly ordered set on S. P_δ is compatible to P_α, and the join of P_α and P_β, written by $P_\alpha \cup P_\beta$.

LEMMA 3. *Let M_α be the totality of the all partly ordered sets which are each other compatible.* For P_α, P_β, $P_\gamma \in M_\alpha$ *P_α is compatible with P_β and P_α is compatible with P_γ then P_β is compatible with P_γ.*

LEMMA 4. $(P_\alpha \cup P_\beta) \cap P_\gamma = (P_\alpha \cap P_\gamma) \cup (P_\beta \cap P_\gamma)$ *in M_α.*

For any $P_\alpha \in M_\alpha$, ($P_\alpha \neq O$, $P_\alpha \neq L_\alpha$), since M_α is an atomic lattice there exist $A(x, y) \leqq P_\alpha$ in M_α if P_α has x, y: $x \leqq y$ in P_α. Let $\mathfrak{A}$ be the set of the all atoms in M_α, and $\mathfrak{A}(P_\alpha)$ be the set of atoms $A(x, y)$ which satisfy $A(x, y) \leqq P_\alpha$. Then $P_\alpha = \cup\{A(x, y) \in \mathfrak{A}(P_\alpha)\}$. For, there exists an element of M_α: $P_\beta = \cup\{A(x, y) \in \mathfrak{A}(P_\alpha)\}$ by completeness of M_α. Obviously $P_\beta \leqq P_\alpha$. If $P_\beta < P_\alpha$ there exists a pair of elements a, b $(a \neq b)$ $a \# b$ in P_β and $a \leqq b$ (or $a \geqq b$) in P_α. This pair a, b determines an atom $A(a, b)$ (or $A(b, a)$) $\in M_\alpha$ because $A(a, b)$ (or $A(b, a)$) $\leqq P_\alpha$. Since $a \# b$ in P_β, $P_\beta \ngeqq A(a, b)$ (or $P_\beta \ngeqq A(b, a)$). This leads a contradiction with $P_\beta = \cup\{A(x, y) \in \mathfrak{A}(P_\alpha)\}$. Thus $P_\alpha = P_\beta$. Let put $P'_\alpha \in M_\alpha$ such that $P'_\alpha = \cup\{A(x, y) \in \mathfrak{A} - \mathfrak{A}(P_\alpha)\}$. Here $\mathfrak{A} - \mathfrak{A}(P_\alpha)$ is nonvoid because in P_α there is a pair c, d: $c \# d$. Thus $\mathfrak{A} - \mathfrak{A}(P_\alpha)$ contains at least one atom $A(c, d)$ (or $A(d, c)$). Then $P_\alpha \# P'_\alpha$ obviously. Hence $P_\alpha \cap P'_\alpha = O$ and $P_\alpha \cup P'_\alpha = L_\alpha$, where L_α is a unit element of M_α. So that we can obtain the following lemma.

LEMMA 5. *M_α is a complemented lattice.*

THEOREM 1. *M_α is an atomic complete Boolean lattice.*

PROOF. By lemma 1 M_α is an atomic complete lattice. Since M_α is distributive by lemma 4. M_α is unique complemented refering to lemma 5. Then M_α is an atomic complete Boolean larttice. [1]

LEMMA 6. *Let M be a lattice in P. Any P_α, P_β belonging to M_α have compatible partial orderings.*

PROOF. For any P_α, P_β of M there exist $P_\alpha \cup P_\beta$ and $P_\alpha \cap P_\beta$ belonging to M. If P_α and P_β have not compatible partial orderings there exists one pair of elements x, y $(x \neq y)$ such that $x \leqq y$ in P_α and $y \leqq x$ in P_β. Then $x \leqq y$ in $P_\alpha \cup P_\beta$ from $x \leqq y$ in P_α, and $y \leqq x$ in $P_\alpha \cup P_\beta$ by $y \leqq x$ in P_β. Thus $x = y$ since $P_\alpha \cup P_\beta$ is a partly ordered set. This is a contradiction with $x \neq y$. Hence P_α and P_β have a compatible partial orderings.

THEOREM 2. *Let M be a maximal lattice in P. Then M has a unit L_α, and zero O. And M is a set of all partly ordered sets P_α which is compatible with L_α in P.*

PROOF. For any partly ordered set P_α in M there is a linearization L_α of P_α. Let M_α be the set of partly ordered sets P_α which are compatible with L_α in P. Then M_α is a lattice. Moreover M_α is a maximal lattice in P. If $M \neq M_\alpha$, there is a partly ordered set $P_\gamma \in M - M_\alpha$ and P_γ is not compatible with L_α. Then $M \ni P_\gamma$, P_α which are not compatible. This means M is not a lattice. Then $M = M_\alpha$.

In P there are a number of linearization of maximal lattices. Let $\mathcal{G}$ be the set of all maximal lattices in P. $\mathcal{G}$ is equivalent to the set of all linearization of S.

THEOREM 3. *$\mathcal{G}$ is equivalent to the set of all linearizations of S.*

THEOREM 4. *Let M_α and M_β are 2 maximal lattices in P. L_α and L_β are the unit elements of M_α and M_β, respectively. If there is an order isomorphism between L_α and L_β or a dual order isomorphism between L_α and L_β. Then M_α is lattice isomorphic to M_β.*

PROOF. If there is an order isomophism τ between L_α and L_β. And a_α of L_α corresponds to a_β of L_β by τ : $\tau(a_\alpha) = a_\beta$. If b_α of L_α corresponds to b_β of L_β : $\tau(b_\alpha) = b_\beta$. When $a_\alpha \leqq b_\alpha$ then $\tau(a_\alpha) \leqq \tau(b_\alpha)$ i.e. $a_\beta \leqq b_\beta$. Thus τ leads a correspondence τ^* between atoms of M_α and atoms of M_β as

$$\tau^* A(a_\alpha, b_\alpha) = A(\tau(a_\alpha), \tau(b_\alpha)) = A(a_\beta, b_\beta).$$

Obviously τ^* is a 1-1 correspondence between the set of all atoms of M_α and the set of all atoms of M_β.

If L_α corresponds to L_β by an order dual isomorphism ϕ. Then ϕ leads the correspondence ϕ^* between atoms in M_α and atoms in M_β, as $\phi^*(A(a_\alpha, b_\alpha)) = A(\phi(b_\alpha), \phi(a_\alpha))$. Thus ϕ^* is also a 1-1 correspondence between the set of all atoms of M_α and the set of all atoms of M_β.

In both cases we obtained a 1-1 correspondence between the set of all atoms of M_α and the set of all atoms of M_β. Since M_α is isomorphic with the Boolean algebra of all subsets of the set of all atoms of M_β, for arbitrary P_α of M_α there exists a subset of the set of all atoms of M_α which satisfies $P_\alpha = \cup\{A(a_\alpha, b_\alpha) \leqq P_\alpha\}$. This is held in M_β, too. We denote the set of atoms correspond with $P_\alpha \in M_\alpha$, $P_\beta \in M_\beta$ by $\mathfrak{A}(P_\alpha)$, $\mathfrak{A}(P_\beta)$, respectively. Then P_α corresponds *to* P_β if and only if $\mathfrak{A}(P_\alpha)$ corresponds with $\mathfrak{A}(P_\beta)$ by correcpondence τ^*(or ϕ^*) :

for $\quad P_\alpha = \cup\{A(a_\alpha, b_\beta) \in \mathfrak{A}(P_\alpha)\}$ and

$$P_\beta = \cup\{A(a_\beta, b_\beta) \in \mathfrak{A}(P_\beta)\},$$

$$\mathfrak{A}(P_\alpha) \longleftrightarrow \mathfrak{A}(P_\beta) \text{ by } \tau^* \text{ (or } \phi^*)$$

$$\Longleftrightarrow P_\alpha \longleftrightarrow P_\beta.$$

And obviously the correspondence between the elements of M_α and the elements of M_β is isomorphism between M_α and M_β from $P_\alpha \leqq P_\gamma$ if and only if $\mathfrak{A}(P_\alpha) \leqq \mathfrak{A}(P_\gamma)$.

COROLLARY. ***There exists a fixed set $\mathfrak{A}$ which satisfies M_α is lattice-isomorphic with $2^{\mathfrak{A}}$ for each α.***

PROOF. To each M_α there is a set of atoms $\mathfrak{A}(M_\alpha)$ which satisfies M_α is lattice isomopphic with $2^{\mathfrak{A}(M_\alpha)}$. The Cardinal number $|\mathfrak{A}(M_\alpha)|$ of $\mathfrak{A}(M_\alpha)$ is equal to $|\mathfrak{A}(M_\beta)|$ for each β because this is the number of combinations of taking 2 from $|S|$. So that we can take a fixed set $\mathfrak{A}$ which satisfies that M_α is lattice-isomorphic with $2^{\mathfrak{A}}$ by correspondence φ_α for each α.

2. The lattice of all partitions on S.

We need a definition about a partly ordered set in this section.
We call the partly ordered set P_α by a *tree* if P_α consist of chains, and in P_α $a \# b$ for any elements a, b belonging distinct chains.

For any fixed set S we are considered with the set B of the partitions of S. In B we introduce a partial ordering $\leqq$ as follows: $\pi_\alpha \leqq \pi_\beta$ for π_α, π_β of B means π_α is a refinement of π_β. Then B is a complete lattice, as well known. We dfine a correspondence between a partition of S and a partly ordered set on S as follows:

for $\pi_\alpha \in B$ $\pi_\alpha = \{\cdots\cdots\cdots, A_\alpha, B_\alpha, \cdots\cdots\cdots\}$ corresponds with
tree $t_\alpha = \{\cdots\cdots, C(A_\alpha), \cdots C(B_\alpha), \cdots\cdots\}$.

Where $C(A_\alpha)$ is a chain orderedset of $A_\alpha \subset S$; if A_α, B_α are components of π_α then for arbitrary x of $C(A_\alpha)$ and arbitrary y of $C(B_\alpha)$ $x \# y$ in t_α; and any t_α, t_β are compatible partial ordered sets each other. We denote the set of this trees by $\mathscr{L}$. Then this correspondence becomes a order homomorphism form $\mathscr{L}$ to B by similar way in the section one. Since the elements of $\mathscr{L}$ are compatible each other this correspondence is 1-1 correspondence. So that there is an order isomorphism between $\mathscr{L}$ and B.

THEOREM 5. *On the fixed set S the lattice of all partitions is lattice isomorphismic with a lattice of all compatible trees on S:*

3. Partly ordered groups.

It is well known that any abstract commutative group whose elements are all of infinite order can be made into a linearly ordered group, [1]

Ladislas Fuchs asserted in his paper that a partial ordering on a commutative group has an extension if the above partial ordering is normal. Moreover he proved that every normal partial ordering on a commutative group has a linear ordering which is an extension of the above one. Here I am concerned with the set G of all normal partly ordered groups on a fixed commutative group A.

A partly ordered group A_α is a commutative group A, written additively, with a relation $<$ which is defined between some pairs of its elements such that the following postulates hold:

(*a*) *any two of the three relations* $a>b, a=b,$ $a<b$ *are contradictory*;

(*b*) *transitivity*: $a>b$ *and* $b>c$ *implies* $a>c$;

(*c*) *homogeneity*: $a>b$ *implies* $a+c>b+c$ *for every* c *in* A;

(*d*) *normality*: $na=a+a+\cdots+a\geqq o$ *for some positive integer* n *implies* $a\geqq 0$.

Conditions (a) and (b) is equivalent with the condition that the relation $\leqq$ is a partial ordering. By the conditions (b) and (c) the relations $a>b,$ $c>d$ may be added to get $a+c>b+d$.

For any A_α of G there exists an extension A_β of A_α which is again a normal partial ordering, [2].

The partial ordering between 2 elements A_α , A_β of G is defined as follows:

$A_\alpha \leqq A_\beta$ if and only if A_β is an extension of A_α .

Let G_1 be any subset of G, $G_1=\{\cdots\cdots, A_\tau ,\cdots\cdots\}$. We define a new partial ordering P on A as follows:

for any 2 elements a,b of A we put $a>b$ in P if and only if $a>b$ in every A_τ in G_1.

It is obviously proved that P is again a partly ordered group. Moreover P is normal if all A_τ in G_1 are normal. The partial order P is said to be the meet of G_1, written by $P=\cap A_\tau$ ($A_\tau \in G_1$). Then G is a meet-complete semilattice.

In this case G conains the commutative group A on which the relation is defined no pair of elements. This is the previous fixed commutative group A. Since A has no element of finite order other than O, identity in commutative group A. A in which no partial order is defined is normal. So we can say that G contains A as the least element.

If we define a normal partial ordering on A as $x<y$ between exact 2 elements x,y of A. By the conditions (a)—(d) of the normal partial ordering on G the fact $x<y$ determine a definite partly ordered group in G such that $y-x>0$; $x+z<y+z$ for arbitrary z of A; and $m(y-x)$ $\geqq n(y-x)$ if n,m are integers $n\leqq m$.

The partly ordered group on A of this type is obviously an atom in G. We denote this by $A(x,y)$.

LEMMA 7. *G is an atomic meet-complete semilattice.*

For a linearly ordered group L_α of A let us put the set N_α of the all normal partly ordered groups $\leqq L_\alpha$. Then N_α is the set of all compatible partly ordered groups and N_α contains a unit element L_α. Hence N_α becomes a complete lattice. Thus we obtain the following lemma.

LEMMA 8. *Let N_α be the set of all normal partly ordered groups $\leqq L_\alpha$, for a linearly ordered group L_α of A. Then N_α becomes an atomic complete lattice.*

In N_α, $A_\alpha \cup A_\beta = A_\gamma$ is a partly ordered group with a normal transitive relation $\leqq$ in A_γ such that

$$x \leqq y \text{ in } A_\alpha \text{ if and only if } x \leqq y \text{ in } A_\alpha \text{ or } x \leqq y \text{ in } A_\beta .$$

Certainly we need the assumption of normality in the above A_γ. And we can assume this assumption without any contradiction. For example if $na \# 0$ in A_α for positive integer n and there exists an element x such that $na \geqq x$ in A_α and $x \geqq 0$ in A_β then $na \geqq 0$ in A_γ by transitivity. But we can not prove that the normality of the relation in A_γ : $a \geqq 0$, eventhough A_α and A_β are normal.

By the similarway in the section one we obtain the followings.

LEMMA 9. *N_α is a distributive lattice.*

LEMMA 10. *For any $A_\alpha \in N_\alpha$ there exists $A_\beta \in N_\alpha$ such that $A_\alpha \cap A_\beta = A$ and $A_\alpha \cup A_\beta = L_\alpha$.*

PROOF. Let $\mathfrak{A}(A_\alpha)$ be the set of all atoms $A(a,b) \leqq A_\alpha$. Then $A_\alpha = \cup \{A(a,b) \in \mathfrak{A}(A_\alpha)\}$. This is proved by the same way in the lemma 5. Now put A_β as the join of all atoms which belong to $\mathfrak{A}(N_\alpha) - \mathfrak{A}(A_\alpha)$ where $\mathfrak{A}(N_\alpha)$ is the set of all atoms in N_α. Then we can easily prove that $A_\alpha \cap A_\beta = A$ and $A_\alpha \cup A_\beta = L_\alpha$, by completeness of N_α, [1]

THEOREM 6. *N_α is an atomic complete Boolean Lattice.*

COROLLARY. *N_α is lattice isomorphic with $2^{\mathfrak{A}(N_\alpha)}$.*

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REFERENCES

〔1〕 G. Birkhoff, *Lattice theory*, rev. ed., New York, (1948).

〔2〕 Ladislas Fuchs, *On the extension of the partial order of groups*, Amer. Journal of Math., Vol. LXXII, No. 1, pp. 191—194, (1950).

〔3〕 E. Szpilrajn, *Sur l'extension de l'ordre partiel*, Fund. Math., Vol.16, pp.386—389, (1930).

〔4〕 $x \# y$ denotes that x is incomparable to y.