NOTES ON THE LATTICE OF SUBGROUPS OF A GROUP

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Introduction. Let G be a group. A set $L(G)$ of all subgroups of G forms a lattice by taking, as the join $X\vee Y$ of X and Y in $L(G)$, a subgroup generated by X and Y, and as the meet $X\vee Y$ a setintersection of X and Y. In this paper, using the representation theorem for partially ordered sets of wolk [1], we shall find that the necessary and sufficient conditions of $L(G)$ to be isomorphic to a set Boolean algebra 2^S for some set S. And finally, we shall show that if G be a group whose $L(G)$ is disjunctive (def. 2), then the necessary and sufficient conditions that $L(G)$ be isomorphic to a lattice of all closed subsets of a T_1-space are that G is a generalized cyclic group.

2. Definitions and preliminaries. We assume that $G=\{e,a,b,\cdots\}$ be a group with an identity e, X, $Y,\cdots$ are subgrouups of G, and $\mathcal{A}$, $\mathcal{L},\cdots$ subsets of $L(G)$. If $\mathcal{A}\subseteq L(G)$, we let

$$\mathcal{A}^{*}=\{X\in L(G)\mid X\geqq A \textit{ for all } A\in\mathcal{A}\}$$
$$\mathcal{A}^{+}=\{X\in L(G)\mid X\leqq A \textit{ for all } A\in\mathcal{A}\}$$

DEFINITION 1 (Frink) A subset $\mathcal{A}$ of $L(G)$ is a *dual ideal* if and only if for every finite subset $\mathcal{F}$ of $\mathcal{A}$, we have $\mathcal{F}^{+*}\subseteq\mathcal{A}$. we call a dual ideal $\mathcal{A}$ *proper* if and only if $\mathcal{A}\neq\{G\}$ and $\mathcal{A}\neq L(G)$. A porper dual ideal which is a proper subset of no proper dual ideal is called *maximal*. For a fixed $X\in L(G)$, the set $\{Y\in L(G)\mid Y\geqq X\}$ is a dual ideal which we call the *principal* dual ideal generated by X and which we denoted by $\mathcal{J}(X)$.

DEFINITION 2 $L(G)$ is disjunctive if and only if for each pair element of X and Y of $L(G)$ with $X\nleqq Y$, there exists $Z\in L(G)$ such that $X\wedge Z\neq\{e\}$ and $Y\wedge Z=\{e\}$. [1].

LEMMA 1 If X is a cyclic subgroup of prime order in G, then $\mathcal{J}(X)$ is a maximal dual ideal in $L(G)$.

PROOF Let $X=\{a\}$. We suppose that $\mathcal{J}(\{a\})\subseteq\mathcal{L}$ for some dual ideal $\mathcal{L}$ of $L(G)$. If $\mathcal{J}(\{a\})\neq\mathcal{L}$, then let A be a subgroup such that $A\notin\mathcal{J}(\{a\})$ and $A\in\mathcal{L}$. Since $\{a\}\nleqq A$ and $\{a\}$ is a minimal subgroup, we see $\{\{a\},A\}^{+}=\{e\}$.
Thus $\{\{a\},A\}^{+*}=\{e\}^{*}(=L(G))\subseteq\mathcal{L}$.

DEFINITION 3 A set $\triangle$ of maximal dual ideals of $L(G)$ is called a *covering family* for $L(G)$ if and only if for each $X(\neq\{e\})\in L(G)$ there exists $\mathfrak{a}\in\triangle$ such that $X\in\mathfrak{a}$ [1].

E.S. Wolk has proved the following representation theorem for partially ordered set:

THEOREM A Partially ordered set P with O and I is isomorphic to 2^s for some set S if and only if

(i) P is disjunctive, and

(ii) there exists a covering family $\triangle$ for P such that $\cap_{M\in\Sigma}M$ is a principal proper dual ideal for every proper nonempty subset Σ of $\triangle$.

3. Main results. At first we consider the relation between the properties of the group G and the fact that $L(G)$ is disjunctive.

PROPOSITION 1 $L(G)$ is disjunctive if and only if the order of any element of G is finite and not divided by any square number.

PROOF Let $L(G)$ be disjunctive. Suppose that there exists an element $a(\neq e)$ of order infinite in G. Let $\{a\}$ be a cyclic subgroup of generated by a, then $\{a^p\}<\{a\}$ for some prime number p. Thus $\{a\}\nleqq\{a^p\}$. Since $L(G)$ is disjunctive, we can find a subgroup A such that $A\wedge\{a\}\neq\{e\}$. And let $A\wedge\{a\}=\{a^r\}$, then $r\neq 0$. And $A\wedge\{a^p\}\geqq\{a^r\}\wedge\{a^p\}=\{a^d\}\neq\{e\}$, where d is the least common multiple of r and p. Hence $A\wedge\{a^p\}\neq\{e\}$ for any A with $A\wedge\{a\}\neq\{e\}$, which means that $L(G)$ is not disjunctive. And suppose that there exists an element a of G whose order is divided by a square number. Let the order of a be $p^\lambda q$ for some prime number p, $(\lambda\geqq 2)$ and $(p,q)=1$. Since $\{a^q\}\nleqq\{a^{pq}\}$, there exists $A\in L(G)$ such that $A\wedge\{a^q\}=\{a^r\}$ $(\neq\{e\})$. Thus q is a factor of r. And $A\wedge\{a^{pq}\}\geqq\{a^r\}\wedge\{a^{pq}\}=\{a^f\}$ where $f=[r,pq]$. But if $p^\lambda q$ is a factor of f, then $p^\lambda q$ is also a factor of r, which is contrary to $\{a^r\}\neq\{e\}$. Hence $\{a^f\}\neq\{e\}$, and $A\wedge\{a^{pq}\}\neq\{e\}$ for any A with $A\wedge\{a^q\}\neq\{e\}$. It is also contrary. Conversely, let A, B are subgroups of G with $A\nleqq B$. And let $a\in A$ and $a\notin B$. By the hypotheses, the order n of a is a finite product of all distinct prime numbers $(=p\cdots r\cdots s)$. But we can take at least one element a of power n/r which is not belonging to B, where r is some prime factor of n. For, otherwise, we have $a\in B$. Because the numbers $n/p, \cdots\cdots, n/s$ are relative prime, by elementary number theory, there exist the numbers q such that $\Sigma q(n/p)=1$. Hence we have the contradiction.

Since the order of the element a of power n/r is prime r, we have $\{a^{n/r}\}\wedge B$

$=\{e\}$. On the other hand, $\{a^{n/r}\}\wedge A=\{a^{n/r}\}$ $(\neq\{e\})$. Hence $L(G)$ is disjunctive.

Clearly, if the order of every element of G is finite, and A is a minimal subgroup of G, then A is a cyclic subgroup of prime order.

DEFINITION 4 $\{A_\lambda\}$ is said to be *subgroup base* for G if and only if $\{A_\lambda\}$ is a minimal family of minimal subgroups such that $\vee A_\lambda=G$.

Then we have the following

PROPOSITION 2 *The lattice of subgroups $L(G)$ is isomorphic to 2^S for some set S if and only if*

(*i*) *the order of any element of G is finite and not divided by any square number*

(*ii*) $\{\{a_\alpha\}|$ *the order of a_α is prime in $G\}$ is a subgroup base for G,*

PROOF Suppose (*i*) and (*ii*) are satisfied in G, By proposition 1 and Wolk's theorem, it is sufficient to show that the condition (ii) of Wolk's theorem is satisfied in $L(G)$. We can take as our covering family $\triangle$ the set of all principal dual ideals generated by $\{a_\alpha\}$ of (*ii*), which are maximal. (see Lemma 1). In fact, let $M\in L(G)$ $(M\neq\{e\})$ and $a\in M$ $(a\neq e)$ of order n. Then the order of element a of power n/p is prime p. (where p is prime factor of n). Thus $M\in\mathcal{J}(\{a^{n/p}\})$. Now let Σ be a proper nonempty subset of $\triangle$. We shall prove that the intersection of $\mathcal{J}(\{a_\alpha\})$ belonging to Σ is a principal proper dual ideal. In fact, we put $H=\vee\{\{a_\alpha\}\mid\mathcal{J}(\{a_\alpha\})\in\Sigma\}$. It is then easily shown that $\mathcal{J}(H)$ is the intersection of $\mathcal{J}(\{a_\alpha\})$ belonging to Σ, that is, a principal dual ideal. Moreover, $\mathcal{J}(H)$ is a poroper. For, if $\mathcal{J}(H)=L(G)$, then $H=\{e\}$ which is contrary. If $\mathcal{J}(H)=G$, then $H=G$, i.e. $G=\vee\{\{a_\alpha\}|\mathcal{J}(\{a_\alpha\})\in\Sigma\}$ which is contrary to that all cyclic groups $\{a_\alpha\}$ of prime order form subgroup base, (but Σ is a proper subset of $\triangle$). Hence the condition (ii) of Wolk's theorem is satisfied. The necessary is obvious. Thus our result follows.

We shall call a group G *generalized* cyclic if any two elements a,b are powers of a suitable third element c of G [2].

COROLLARY 1 *Let G satisfy* (*i*) *and* (*ii*) *of propcsiticn 2, then G is a generalized cyclic group.*

If G is a finite, then the condition (*ii*) of proposition is unnecessary.

PROPOSITION 3 *Let G be a finite grcup. $L(G)$ is isomorphic to for some finite set S if and only if G is cyclic group whose order is not divided by any square number.*

PROOF The necessary is obvious by corollary 1. Let G be a cyclic group $\{a\}$ of order n. By hypotheses the prime factors of n are all distinct. Thus we can easily see that

$$\{\{a^i\} \mid \text{the order of } a^i \text{ is prime}\} = \{\{a^{n/p}\} \mid p \text{ is prime factor of } n\}.$$

And since the numers n/p are relative prime, we see $\vee\{\{a^{n/p}\}\}=G$. But since $L(G)$ is distributive lattice (see [2]. p. 96), $\{a^{n/q}\}\wedge\{\vee_{p\neq q}\{a^{n/p}\}\}=\{e\}$. Hence $\{\{a^{n/p}\} \mid p$ is prime factor of $n\}$ is a subgroup base for G. Therefore the proof is complete.

4. Another representation of disjunctive $L(G)$. E. S. Wolk has also proved the following [1].

THEOREM *Let L be a complete, atomic, and disjunctive lattice, and S its set of points. A necessary and sufficient condition that L be isomorphic to the lattice of all closed subsets of a T_1-space X (where X is in 1:1 correspondence with S) is that $P \in S$ and $p \leqq a \vee b$ implies $p \leqq a$ or $p \leqq b$.*

Clearly, we see that if $L(G)$ is disjunctive, then $L(G)$ is complete, atomic lattice. Hence if the word "lattice L" in above throrem is everywhere replaced by "lattice $L(G)$", then our result may be stated in the form

PROPOSITION 4 *Let G be a group whose $L(G)$ is disjunctive. A necessary and sufficient condition that $L(G)$ be isomorphic to the lattice of all closed subsets of a T_1-space X (where X is in 1:1 correspondence with $\{\{a_\alpha\} \mid$ the order of a_α is prime$\}$) are that G is a generatized cyclic group.*

The proof of this proposition is directly given by the following Lemma and Ore's theorem ([2] p. 96).

LEMMA 2 *Let G be a group whose $L(G)$ is disjunctive. $L(G)$ is distributive if and only if for any cyclic subgroup $\{a\}$ of order prime, if $\{a\} \leqq A \vee B$ for some $A, B \in L(G)$, then $\{a\} \leqq A$ or $\{a\} \leqq B$.*

PROOF Suppose $L(G)$ is distributive. If $\{a\} \leqq A \vee B$, then $\{a\} = (\{a\} \wedge A) \vee (\{a\} \wedge B)$. Thus if $\{a\} \wedge A = \{e\}$, then $\{a\} \leqq B$. Conversely, Suppose that the conditions are satisfied in $L(G)$. By the one-side distributive law, $L(G)$ is distributive if $C \wedge (A \vee B) \leqq (C \wedge A) \vee (C \wedge B)$ for any three subgroups A, B, C of G. Let $a (\neq e) \in C \wedge (A \vee B)$. Then $a \in C$ and $a \in A \vee B$. Since element a of power n/p is in $A \vee B$ for any prime factor p of n (where n is the order of a), it is in A or B by hypotheses, i.e. element a of power n/p is in $A \wedge C$ or $B \wedge C$ for any prime factor p of n. But since all numbers n/p (p is prime factor of n) are relative prime, we can choose the numbers q such that $\Sigma q(n/p)=1$. Thus we have $a \in (A \wedge C) \vee (B \wedge C)$.

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REFERENCES

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