SYMMETRIC SPACES WHICH ARE MAPPED CONFORMALLY ON EACH OTHER

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Introduction.

The considerations of Einstein spaces, which are mapped conformally on each other, were studied by Brinkmann [1]. In this paper we are going to study about symmetric spaces as in the case of Einstein spaces.

Let V_n and $\bar{V}_n$ be n-dimensional Riemannian spaces with their fundamental metric tensors g_{ij} and $\bar{g}_{ij}$ respectively, and the correspondence between them is given by

(0. 1) $\bar{g}_{ij}=e^{2\sigma}g_{ij}, \qquad \sigma=\sigma(x^1, \cdots\cdots, x^n)$,

then, we have the following relations [2]:

(0. 2) $\bar{g}^{ij}=e^{-2\sigma}g^{ij}$,

(0. 3) $\overline{\{{}_i{}^l{}_j\}}=\{{}_i{}^l{}_j\}+\delta^l_i\sigma_{,j}+\delta^l_j\sigma_{,i}-g_{ij}g^{lm}\sigma_{,m}$,

where $\sigma_{,i}=\dfrac{\partial\sigma}{\partial x^i}$. If $\sigma_{,ij}$ denote the second covariant derivatives of σ with respect to g_{ij}, and we put

(0. 4) $\sigma_{ij}=\sigma_{,ij}-\sigma_{,i}\sigma_{,j}$,

then we have

(0. 5) $$e^{-2\sigma}\bar{R}_{hijk}=R_{hijk}+g_{hk}\sigma_{ij}+g_{ij}\sigma_{hk}-g_{hj}\sigma_{ik}-g_{ik}\sigma_{hj}+(g_{hk}g_{ij}-g_{hj}g_{ik})\Delta_1\sigma ,$$

where

$$\Delta_1\sigma=g^{ij}\sigma_{,i}\sigma_{,j} .$$

Furthermore we have

(0. 6) $\bar{R}_{ij}=R_{ij}+(n-2)\sigma_{ij}+[\Delta_2\sigma+(n-2)\Delta_1\sigma]g_{ij}$,

where

$$\Delta_2\sigma=g_{ij}\sigma_{,ij} ,$$

and

(0. 7) $\bar{R}=e^{-2\sigma}[R+2(n-1)\Delta_2\sigma+(n-1)(n-2)\Delta_1\sigma]$,

$$(0.8) \qquad \sigma_{ij}=\frac{1}{(n-2)}(\bar{R}_{ij}-R_{ij})-\frac{1}{2(n-1)(n-2)}(\bar{g}_{ij}\bar{R}-g_{ij}R) -\frac{1}{2}g_{ij}\Delta_1\sigma.$$

1. Some Preliminaries.

Let us think about an n-dimensional Riemannian space V_n with the metric tensor g_{ij}, and an n-dimensional symmetric space $\bar{V}_n$ with the metric tensor $\bar{g}_{ij}$, which are related to g_{ij} in the equations (0.1). For any second order covariant tensor B_{ij}, we have the following relations in virtue of (0.3)

$$(1.1) \qquad B_{ij|l}=B_{ij,l}-2B_{ij}\sigma_{,l}-B_{lj}\sigma_{,i}-B_{il}\sigma_{,j} +(g_{il}B_{aj}+g_{jl}B_{ia})\, g^{am}\sigma_{,m},$$

where comma (,) and solidus (|) denote the covariant derivatives with respect to g_{ij} and $\bar{g}_{ij}$ respectively. In the same manner, for any second order contravariant tensor B^{ij}, we have

$$(1.2) \qquad B^{ij}{}_{|l}=B^{ij}{}_{,l}+2B^{ij}\sigma_{,l}+(\delta^i_l B^{aj}+\delta^j_l B^{ia})\,\sigma_{,a} -(g^{im}B^{aj}+g^{jm}B^{ia})\,g_{al}\sigma_{,m}.$$

Generally, we can have the following relations between the two kind of covariant derivatives (,) and (|) for any tensor $T^{ab\cdots cd}{}_{ki\cdots jk}$ whose contravariant order is p and covariant orer is q

$$(1.3) \qquad \begin{aligned} T^{ab\cdots cd}{}_{ki\cdots jk|l}=&T^{ab\cdots cd}{}_{ki\cdots jk,l}+(p-q)T^{ab\cdots cd}{}_{ki\cdots jk}\sigma_{,l}\\ &+(\delta^a_l T^{nb\cdots cd}{}_{ki\cdots jk}+\cdots+\delta^d_l T^{ab\cdots cn}{}_{ki\cdots jk})\sigma_{,n}\\ &-(g^{am}T^{nb\cdots cd}{}_{ki\cdots jk}+\cdots+g^{dm}T^{ab\cdots cn}{}_{ki\cdots jk})g_{nl}\sigma_{,m}\\ &-(\sigma_{,k}T^{ab\cdots cd}{}_{li\cdots jk}+\cdots+\sigma_{,k}T^{ab\cdots cd}{}_{ki\cdots jl})\\ &+(g_{kl}T^{ab\cdots cd}{}_{ni\cdots jk}+\cdots+g_{kl}T^{ab\cdots cd}{}_{ki\cdots jn})g^{nm}\sigma_{,m} \end{aligned}$$

The above-mentioned general rule gives us the following relations immediately:

$$(1.4) \qquad g_{ij|l}=-2g_{ij}\sigma_{,l}, \quad g^{ij}{}_{|l}=2g^{ij}\sigma_{,l},$$

$$(1.5) \qquad \delta^i_{j|l}=0,$$

and

$$(1.6) \qquad R_{ij|l}=R_{ij,l}-2R_{ij}\sigma_{,l}-R_{il}\sigma_{,j}-R_{jl}\sigma_{,i} +(g_{il}R^a{}_j+g_{lj}R^a{}_i)\,\sigma_{,a}.$$

Furthermore, on account of (0.8), we have

$$\sigma_{ij|l}=-\frac{1}{n-2}R_{ij|l}+\frac{1}{2(n-1)(n-2)}[R_{|l}-2R\sigma_{|l} \tag{1.7}$$
$$+2(n-1)(n-2)(\Delta_1\sigma)\,\sigma_{|l}$$
$$-(n-1)(n-2)(\Delta_1\sigma)_{|l}]g_{ij}\ ,$$

which can be rewritten in the following form by means of (1.6)

$$\sigma_{ij|l}=-\frac{1}{n-2}[R_{ij,l}-2R_{ij}\sigma_{,l}-R_{lj}\sigma_{,i}-R_{il}\sigma_{,j}$$
$$+(g_{il}R^a{}_j+g_{lj}R^a{}_i)\sigma_{,a}]+\frac{1}{2(n-1)(n-2)}[R_{,l} \tag{1.8}$$
$$-2R\sigma_{,l}+2(n-1)(n-2)(\Delta_1\sigma)\sigma_{,l}$$
$$-(n-1)(n-2)(\Delta_1\sigma)_{,l}]g_{ij}\ .$$

We have used $R_{,l}$ and $(\Delta_1\sigma)_{,l}$ instead of $R_{|l}$ and $(\Delta_1\sigma)_{|l}$ in the above equations (1.8). In fact, for the scalar function, the covariant derivatives with respect to g_{ij} are equal to the one with respect to g_{ij}.

Multiplying g^{ij} to (1.8) and contracting over i and j, we have

$$g^{ij}\sigma_{ij|l}=\frac{1}{2(n-1)}(2R\sigma_{,l}-R_{,l}) \tag{1.9}$$
$$+n[(\Delta_1\sigma)\sigma_{,l}-\frac{1}{2}(\Delta_1\sigma)_{,l})]\ .$$

2. Conformal correspondence of symmetric spaces.

In order to think about the conformal correspondence of symmetric spaces, the conditions that a Riemannian space can be mapped conformally on a symmetric space are desired to find out.

Let V_n and $\bar{V}_n$ be n-dimensional spaces mentioned in the previous section, since $\bar{V}_n$ is a symmetric space, we have

$$\bar{R}_{hijk|l}=e^{2\sigma}[R_{hijk|l}+g_{hk|l}\sigma_{ij}+g_{ij|l}\sigma_{hk}-g_{hj|l}\sigma_{ik}$$
$$-g_{ik|l}\sigma_{hj}+g_{hk}\sigma_{ij|l}+g_{ij}\sigma_{hk|l}-g_{hj}\sigma_{ik|l}-g_{ik}\sigma_{hj|l}$$
$$+(g_{hk|l}g_{ij}+g_{hk}g_{ij|l}-g_{hj|l}g_{ik}-g_{hj}g_{ik|l})\Delta_1\sigma \tag{2.1}$$
$$+(g_{hk}g_{ij}-g_{hj}g_{ik})(\Delta_1\sigma)_{|l}]+2e^{2\sigma}[R_{hijk}+g_{hk}\sigma_{ij}$$
$$+g_{ij}\sigma_{hk}-g_{hj}\sigma_{ik}-g_{ik}\sigma_{hj}+(g_{hk}g_{ij}-g_{hj}g_{ik})(\Delta_1\sigma)]\ \sigma_{|l}$$
$$=0\ .$$

On account of (1.3), (1.4) and (0.8), (2.1) is as follows:

$$
\begin{aligned}
g_{hk}\sigma_{ij|l} = & \frac{1}{2}\left\{\frac{e^{2\sigma}\bar{R}}{(n-1)(n-2)}+\Delta_1\sigma\right\}g_{hk}g_{ij}\sigma_{|l} \\
& +\frac{1}{n-2}(R_{hk|l}g_{ij}-R_{ik|l}g_{hj}-R_{hj|l}g_{ik}) \\
& -\frac{1}{n-2}(e^{2\sigma}\bar{R}^a{}_l-R^a{}_l)g_{hk}g_{ij}\sigma_{|a} \qquad (2.2)\\
& +\frac{R}{2(n-1)(n-2)}(g_{hk}g_{ij}-4g_{hj}g_{ik})\sigma_{|l} \\
& -\frac{R_{|l}}{2(n-1)(n-2)}(g_{hk}g_{ij}-g_{hj}g_{ik}) \\
& -R_{hijk|l}-R_{hijk}\sigma_{|l}\ .
\end{aligned}
$$

By putting

$$
\begin{aligned}
L_{hijk} = & \frac{1}{2}\left\{\frac{e^{2\sigma}\bar{R}}{(n-1)(n-2)}+\Delta_1\sigma\right\}g_{hk}g_{ij} \qquad (2.3)\\
& +\frac{R}{2(n-1)(n-2)}(g_{hk}g_{ij}-g_{hj}g)_{ik}-R_{hijk}\ ,
\end{aligned}
$$

$$
\begin{aligned}
\Lambda_{hijkl} = & \frac{1}{n-2}\{R_{hk|l}g_{ij}-R_{ik|l}g_{hj}-R_{hj|l}g_{ik} \qquad (2.4)\\
& -(e^{2\sigma}\bar{R}^a{}_l-R^a{}_l)\sigma_{|a}g_{hk}g_{ij}\}-R_{hijk|l}\ ,
\end{aligned}
$$

(2.2) can be reducible as

$$
\begin{aligned}
g_{hk}\sigma_{ij|l} = & L_{hijk}\sigma_{|l}+\Lambda_{hijkl} \qquad (2.5)\\
& -\frac{R_{|l}}{2(n-1)(n-2)}(g_{hk}g_{ij}-g_{hj}g_{ik})\ .
\end{aligned}
$$

Hence we have the following theorem, because the sufficiency of the theorem is evident, since we did not contract (2.5) with respect to any indices.

THEOREM 1. *A necessary and sufficient condition that a* V_n *for* $n>2$ *can be mapped conformally on a symmetric space* $\bar{V}_n$ *is that there exist a function* σ *satisfying the equations* (2.5).

Now let us find the integrability conditions of the equations (2.5). On account of (1.5), we have the following equations by multiplying

g^{sk} to (2.2)

$$
\begin{aligned}
(2.6)\qquad \delta^s_k\sigma_{;j|l} = & \frac{1}{2}\Big\{\frac{e^{2\sigma}\overline{R}}{(n-1)(n-2)} + \Delta_1\sigma\Big\}\delta^s_k g_{ij}\sigma_{|l} \\
& + \frac{1}{n-2}(R^s{}_{k|l}\, g_{ij} - 2R^s{}_k g_{ij}\sigma_{|l} - \delta^s_j R_{ik|l} - R^s{}_{j|l}\, g_{ik} \\
& + 2R^s{}_j g_{ik}\sigma_{|l}) - \frac{1}{n-2}(e^{2\sigma}\,\overline{R}^a{}_l - R^a{}_l)\,\delta^s_k g_{ij}\sigma_{|a} \\
& + \frac{R}{2(n-1)(n-2)}(\delta^s_k g_{ij} - 4\delta^s_j g_{ik})\,\sigma_{|l} \\
& - \frac{R_{|l}}{2(n-1)(n-2)}(\delta^s_k g_{ij} - 2\delta^s_j g_{ik}) \\
& - R^s{}_{ijk|l} + R^s{}_{ijk}\,\sigma_{|l}\ .
\end{aligned}
$$

By effecting the covariant derivatives with respect to $\overline{g}^{ij}$, we have

$$
\begin{aligned}
(2.7)\qquad \delta^s_k\,\sigma_{ij|lm} - \delta^s_k\,\sigma_{ij|ml} = & \frac{1}{2}\{(\Delta_1\sigma)_{|m}\sigma_{|l} - (\Delta_1\sigma)_{|l}\sigma_{|m}\}\delta^s_k g_{ij} \\
& + \frac{1}{n-2}[(R^s{}_{k|lm} - R^s{}_{k|ml})g_{ij} - \delta^s_j(R_{ik|lm} - R_{ik|ml}) \\
& - (R^s{}_{j|lm} - R^s{}_{j|ml})g_{ik}] + \frac{1}{n-2}(R^a{}_{l|m} - R^a{}_{m|l})\delta^s_k\, g_{ij}\sigma_{|a} \\
& - \frac{2}{n-2}(R^a{}_l\sigma_{|m} - R^a{}_m\,\sigma_{|l})\,\delta^s_k\, g_{ij}\,\sigma_{|a} \\
& - \frac{e^{2\sigma}}{n-2}(\overline{R}^a{}_l\sigma_{|am} - \overline{R}^a{}_m\,\sigma_{|al})\delta^s_k\, g_{ij} \\
& + \frac{1}{n-2}(R^a{}_l\,\sigma_{|am} - R^a{}_m\,\sigma_{|al})\delta^s_k\, g_{ij} \\
& + \frac{1}{2(n-1)(n-2)}(R_{|l}\sigma_{|m} - R_{|m}\sigma_{|l})\,\delta^s_k\, g_{ij} \\
& - (R^s{}_{ijk|lm} - R^s{}_{ijk|ml}) + (R^s{}_{ijk|m}\,\sigma_{|l} - R^s{}_{ijk|l}\,\sigma_{|m})\ .
\end{aligned}
$$

In virtue of Ricci identity, we have the following integrability conditions of the equations (2.5) :

$$
(2.8)\qquad K^s{}_{ijklm} + \delta^s_k B_{lm} g_{ij} = T^s{}_{ijklm}\ ,
$$

where we have put

$$
B_{lm} = \frac{1}{2}\{(\Delta_1\sigma)_{|m}\sigma_{|l} - (\Delta_1\sigma)_{|l}\sigma_{|m}\}
$$

$$+\frac{1}{n-2}(R^a{}_{l|m}-R^a{}_{m|l})\,\sigma_{|a}$$

$$(2.9)\qquad -\frac{2}{n-2}(R^a{}_l\sigma_{|m}-R^a{}_m\sigma_{|l})\sigma_{|a}-\frac{e^{2\sigma}}{n-2}(\bar{R}^a{}_l\sigma_{|am}$$

$$-\bar{R}^a{}_m\sigma_{|al})+\frac{1}{n-2}(R^a{}_l\sigma_{a|m}-R^a{}_m\sigma_{|al})\ ,$$

$$K^s{}_{ijklm}=\frac{1}{n-2}[(R^s{}_{k|lm}-R^s{}_{k|ml})g_{ij}-\delta^s_j(R_{ik|lm}-R_{ik|ml})$$

$$(2.10)\qquad -(R^s{}_{j|lm}-R^s{}_{j|ml})\,g_{ik}]-(R^s{}_{ijk|lm}-R^s{}_{ijk|ml})$$

$$+(R^s{}_{ijk|m}\sigma_{|l}-R^s{}_{ijk|l}\,\sigma_{|m})\ ,$$

$$(2.11)\qquad T^s{}_{ijklm}=\delta^s_k(\sigma_{hj}\,\bar{R}^h{}_{ilm}+\sigma_{ih}\,\bar{R}^h{}_{jlm})\ .$$

Hence we have the following

THEOREM 2. *The integrability conditions of the equations* (2.5) *are given by the equations* (2.8).

Next, we shall find out the conditions that two symmetric spaces can be in conformal correspondence. We can easily obtain them by substituting $R_{|l}=R_{,l}=0$ into the equations (2.5), hence we have the following

THEOREM 3. *In order that a symmetric space can be mapped conformally on a symmetric space, it is necessary that the function* σ *satisfies the equations*

$$(2.12)\qquad g_{hk}\sigma_{ij|l}=L_{hijk}\sigma_{|l}+\Lambda_{hijkl}\ .$$

In virtue of (1.6), (2.12) is written as follows:

$$(2.13)\qquad g_{hk}\,\sigma_{ij|l}=L_{hijk}\,\sigma_{|l}-\frac{1}{n-2}(e^{2\sigma}\bar{R}^a{}_l-R^a{}_l)g_{hk}g_{ij}\sigma_{,a}$$

$$+\frac{1}{n-2}[\{(g_{hl}R^a{}_k+g_{lk}R^a{}_h)\sigma_{,a}$$

$$-2R_{hk}\sigma_{,l}-R_{lk}\sigma_{,h}-R_{hl}\sigma_{,k}\}g_{ij}+\{2R_{hj}\sigma_{,l}$$

$$+R_{lj}\sigma_{,h}+R_{hl}\sigma_{,j}+(g_{hl}R^a{}_j+g_{lj}R^a{}_h)\sigma_{,a}\}g_{ik}$$

$$+\{2R_{ik}\sigma_{,l}+R_{lk}\sigma_{,i}+R_{il}\sigma_{,k}+(g_{il}R^a{}_k$$

$$+g_{ik}R^{a}{}_{i})\sigma,_{a}\}g_{hj}]+4R_{hijk}\sigma,_{l}+R_{lijk}\sigma,_{h}+R_{hljk}\sigma,_{i}$$
$$+R_{hilk}\sigma,_{j}+R_{hijl}\sigma,_{k}-(g_{hl}R^{a}{}_{ijk}+g_{il}R_{h}{}^{a}{}_{jk}$$
$$+g_{jl}R_{hi}{}^{a}{}_{k}+g_{kl}R_{hij}{}^{a})\,\sigma,_{a}\ .$$

Finally, we shall find out a condition that both of the scalar curvatures of V_n and $\bar{V}_n$ vanish.

Multiplying g^{hk} and g^{ij} to (2.13) and summing over h, k, i, and j, we have

$$g^{ij}\sigma_{ij|l}=\frac{n}{2}\left\{\frac{e^{2\sigma}\bar{R}}{(n-1)(n-2)}+\Delta_1\sigma\right\}\sigma,_{l}$$

$$(2.14)\qquad -\frac{n}{(n-2)}(e^{2\sigma}\bar{R}^{a}{}_{l}-R^{a}{}_{l})\,\sigma,_{a}$$

$$+\frac{8n^3-29n^2+14n-12}{2n(n-1)(n-2)}R\sigma,_{l}\ ,$$

and on account of (1.9), the equations (2.14) can be reducible as follows:

$$\left(\frac{\Delta_1\sigma}{e^{\sigma}}\right),_{l}=\frac{2}{n-2}(e^{\sigma}\bar{R}^{a}{}_{l}+e^{-\sigma}R^{a}{}_{l})\sigma,_{a}$$

$$(2.15)\qquad -\frac{\bar{R}}{(n-1)(n-2)}(e^{\sigma}),_{l}$$

$$+\frac{8n^3-31n^2+18n-12}{n^2(n-1)(n-2)}R(e^{-\sigma}),_{l}\ .$$

If $\Delta_1\sigma$ satisfy the following equations:

$$(2.16)\qquad \left(\frac{\Delta_1\sigma}{e^{\sigma}}\right),_{l}=\frac{2}{n-2}(\bar{R}^{a}{}_{l}e^{\sigma}+R^{a}{}_{l}e^{-\sigma})\ \sigma,_{a}$$

we have

$$(2.17)\qquad \frac{\bar{R}}{(n-1)(n-2)}e^{2\sigma}-\frac{8n^3-31n^2+18n-12}{n^2(n-1)(n-2)}R+ce^{\sigma}=0\ ,$$

where "c" is constant.

On the other hand, a necessary and sufficient condition that there exists $\Delta_1\sigma$ satisfying the equations (2.16) is

$$(2.18)\qquad R^{a}{}_{[l}\sigma_{m]a}=\bar{R}^{a}{}_{[l}\sigma_{m]a}=0\ ,$$

since

$$(R^a{}_l e^{-\sigma}\sigma_{,a})_{,m} - (R^a{}_m e^{-\sigma}\sigma_{,a})_{,l} = e^{-\sigma}(R^a{}_l\sigma_{am} - R^a{}_m\sigma_{al}) \ ,$$

and

$$(e^{\sigma}\bar{R}^a{}_l\sigma_{,a})_{,m} - (e^{\sigma}\bar{R}^a{}_m\sigma_{,a})_{,l} = \bar{R}^a{}_{[l}\sigma_{m]a} \ .$$

From the equation (2.17), $R=\bar{R}=0$ if $\Delta_1\sigma$ satisfy the equations (2.16), hence we have the following

THEOREM 4. *If $\Delta_1\sigma$ satisfy the equations (2.18), both of the scalar curvatures of the two spaces V_n and $\bar{V}_n$ vanish.*

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