DIRECT COMPLEXIFICATION OF A RIEMANNIAN MANIFOLD INTO A KAEHLERIAN MANIFOLD

By Sang-Seup Eum

1. Introduction.

We consider a complex analytic n-dimensional manifold whose complex analytic structure $(z^{\alpha}, z^{\bar{\alpha}})$ $(z^{\bar{\alpha}}=\bar{z}^{\alpha}; \alpha, \beta, \gamma\cdots=1,2,\cdots,n;\ \bar{\alpha},\bar{\beta},\bar{\gamma},\cdots=\bar{1},\bar{2},\cdots,\bar{n})$ was given by the relations [1], [2]

(1. 1) $$z^{\alpha}=x^{\alpha}+ix^{\bar{\alpha}},\qquad z^{\bar{\alpha}}=x^{\alpha}-ix^{\bar{\alpha}},$$

where $(x^{A})=(x^{\alpha}, x^{\bar{\alpha}})$ $(A, B, C,\cdots=1,2,\cdots,n, \bar{1},\bar{2},\cdots,\bar{n})$ is a system of coordinate neighborhoods of a real $2n$-dimensional manifold X_{2n}.

By considering that (1.1) is a transformation of coordinates, the real Riemannian metric ds^2 in X_{2n} is related by

(1. 2) $$ds^2=b_{AB}dx^{A}dx^{B}=g_{AB}dz^{A}dz^{B},\quad g_{AB}(z)=\frac{\partial x^{C}}{\partial z^{A}}\frac{\partial x^{D}}{\partial z^{B}}b_{CD}(x).$$

If the metric tensor g_{AB} is hybrid and self-adjoint, i.e.

(1. 3) $$g_{AB}=(0,\ g_{\alpha\bar{\beta}},\ g_{\bar{\alpha}\beta},\ 0),\qquad g_{\alpha\bar{\beta}}=g_{\bar{\beta}\alpha}=\overline{g_{\bar{\alpha}\beta}}=\overline{g_{\beta\bar{\alpha}}}$$

then the metric form (1.2) can be written in the form

(1. 4) $$ds^2=2g_{\alpha\bar{\beta}}dz^{\alpha}d\bar{z}^{\beta}$$

and a metric (1.4) satisfying (1.3) is called a Hermitian metric, and above complex analytic manifold to which a Hermitian metric was given, is called a Hermitian manifold H_n.

For the real Riemannian manifold X_{2n}, we denote the metric tensor, Christoffel symbols, and curvature tensor by the notations b_{AB}, $\{{}^{A}_{BC}\}$ and $R^{A}{}_{BCD}$ respectivery, and for the above Hermitian manifold H_n, by g_{AB}, Γ^{A}_{BC} and $K^{A}{}_{BCD}$ respectively, then we have $\Gamma^{\alpha}_{\bar{\beta}\bar{\gamma}}=0$ (conj.). Moreover if a Hermitian metric satisfies the so-called Kaehlerian condition $\Gamma^{\alpha}_{\bar{\beta}\gamma}=0$ (conj.), then it is called a Kaehlerian metric, and the complex analytic manifold with a Kaehlerian metric is called a Kaehlerian manifold K_n[2].

The main purpose of this paper is characterization of the relations of the geometric objects between a real manifold X_{2n} and a Kaehlerian manifold K_n by means of the relations [3]

$$(1.5)\qquad \frac{\partial}{\partial z^\alpha}=\frac{1}{2}\left(\frac{\partial}{\partial x^\alpha}-i\frac{\partial}{\partial x^{\bar\alpha}}\right),\qquad \frac{\partial}{\partial \bar z^\alpha}=\frac{1}{2}\left(\frac{\partial}{\partial x^\alpha}+i\frac{\partial}{\partial x^{\bar\alpha}}\right),$$

$$\frac{\partial}{\partial x^\alpha}=\frac{\partial}{\partial z^\alpha}+\frac{\partial}{\partial \bar z^\alpha},\qquad \frac{\partial}{\partial x^{\bar\alpha}}=i\left(\frac{\partial}{\partial z^\alpha}-\frac{\partial}{\partial \bar z^\alpha}\right),$$

under the transformation (1.1).

2. Main relations.

By direct calculation from (1.2) using (1.5), we get thg following results

$$g_{\alpha\beta}(z)=\frac{1}{4}[b_{\alpha\beta}(x)-b_{\bar\alpha\bar\beta}(x)]+\frac{i}{4}[b_{\alpha\bar\beta}(x)+b_{\bar\alpha\beta}(x)],$$

$$g_{\alpha\bar\beta}(z)=\frac{1}{4}[b_{\alpha\beta}(x)+b_{\bar\alpha\bar\beta}(x)]+\frac{i}{4}[b_{\alpha\bar\beta}(x)-b_{\bar\alpha\beta}(x)],$$

$$g_{\bar\alpha\beta}(z)=\frac{1}{4}[b_{\alpha\beta}(x)+b_{\bar\alpha\bar\beta}(x)]-\frac{i}{4}[b_{\alpha\bar\beta}(x)-b_{\bar\alpha\beta}(x)],$$

$$g_{\bar\alpha\bar\beta}(z)=\frac{1}{4}[b_{\alpha\beta}(x)-b_{\bar\alpha\bar\beta}(x)]-\frac{i}{4}[b_{\alpha\bar\beta}(x)+b_{\bar\alpha\beta}(x)],$$

then, the tensor g_{AB} is self-adjoint.

If the metric tensor g_{AB} is hybrid, we have

$$(2.1)\qquad b_{\alpha\beta}=b_{\bar\alpha\bar\beta},\qquad b_{\alpha\bar\beta}=-b_{\bar\alpha\beta},$$

and conversely. Then we get the follwing

THEOREM 2.1. *A necessary and sufficient condition that the metric tensor g_{AB} which is related by (1.2) to the real Riemannian metric tensor b_{AB} be a Hermitian one, is the matrix (b_{AB}) is the following form:*

$$(b_{AB})=\begin{pmatrix} P & Q\\ -Q & P\end{pmatrix}$$

where the $(n\times n)$ matrix $P=(b_{\alpha\beta})$ is symmetric, and the $(n\times n)$ matrix $Q=(b_{\alpha\bar\beta})$ is skew-symmetric.

Therefore the Hermitian metric tensor is as follows

(2. 2)
$$g_{\alpha\bar{\beta}}=\frac{1}{2}(b_{\alpha\beta}+ib_{\alpha\bar{\beta}}), \qquad (conj.)$$
$$g^{\alpha\bar{\beta}}=2(b^{\alpha\beta}-ib^{\alpha\bar{\beta}}), \qquad (conj.)$$

where g^{AB} is also self-adjoint, and

$$b^{\alpha\beta}=b^{\bar{\alpha}\bar{\beta}}, \qquad b^{\alpha\bar{\beta}}=-b^{\bar{\alpha}\beta},$$
$$g_{\alpha\bar{\beta}}g^{\bar{\beta}\gamma}=\delta^{\gamma}_{\alpha}, \qquad b_{AB}b^{BC}=\delta^{C}_{A}.$$

By using (1.5) and (2.1), the Kaehlerian condition $\Gamma^{\alpha}_{\beta\bar{\gamma}}=0$ become equivalent to

$$\frac{\partial b_{\alpha\beta}}{\partial x^{\gamma}}-\frac{\partial b_{\bar{\alpha}\beta}}{\partial x^{\bar{\gamma}}}=\frac{\partial b_{\gamma\beta}}{\partial x^{\alpha}}-\frac{\partial b_{\bar{\gamma}\beta}}{\partial x^{\bar{\alpha}}},$$

$$\frac{\partial b_{\alpha\beta}}{\partial x^{\bar{\gamma}}}+\frac{\partial b_{\bar{\alpha}\beta}}{\partial x^{\gamma}}=\frac{\partial b_{\gamma\beta}}{\partial x^{\bar{\alpha}}}+\frac{\partial b_{\bar{\gamma}\beta}}{\partial x^{\alpha}},$$

and by calculating the Christoffel symbols under the Kaehlerian condition, we have

(2. 3) $$[\gamma,\alpha\beta,]-[\bar{\gamma},\bar{\alpha}\beta]=0, \qquad [\gamma,\bar{\alpha}\beta]+[\bar{\gamma},\alpha\beta]=0,$$

where

$$[C,AB]=\frac{1}{2}\left(\frac{\partial b_{BC}}{\partial x^{A}}+\frac{\partial b_{AC}}{\partial x^{B}}-\frac{\partial b_{AB}}{\partial x^{C}}\right),$$

and then

(2. 4) $$\left\{{\alpha \atop \beta\gamma}\right\}=\left\{{\bar{\alpha} \atop \beta\bar{\gamma}}\right\}=\left\{{\bar{\alpha} \atop \bar{\beta}\gamma}\right\}=-\left\{{\alpha \atop \bar{\beta}\bar{\gamma}}\right\}, \quad \left\{{\alpha \atop \beta\bar{\gamma}}\right\}=\left\{{\alpha \atop \bar{\beta}\gamma}\right\}=-\left\{{\bar{\alpha} \atop \beta\gamma}\right\}=\left\{{\bar{\alpha} \atop \bar{\beta}\bar{\gamma}}\right\},$$

where

$$\left\{{A \atop BC}\right\}=b^{AD}[D,BC]$$

Hence we have the following

THEOREM 2.2. *A necessary and sufficient condition that the metric related by* (1.2) *be a Kaehlerian metric, is that* b_{AB} *and* $\left\{{A \atop BC}\right\}$ *are the forms* (2.1) *and* (2.4) *respectively.*

Therefore, the Christoffel symbols in the Kaehlerian manifold K_n are the form

(2. 5) $$\Gamma^{\alpha}_{\beta\gamma}=\left\{{\alpha \atop \beta\gamma}\right\}-i\left\{{\alpha \atop \beta\bar{\gamma}}\right\}, \qquad (conj.)$$

i.e.

(2. 6) $$\left\{{\alpha \atop \beta\gamma}\right\}=\frac{1}{2}(\Gamma^{\alpha}_{\beta\gamma}+\overline{\Gamma^{\alpha}_{\beta\gamma}}), \qquad \left\{{\alpha \atop \beta\bar{\gamma}}\right\}=\frac{i}{2}(\Gamma^{\alpha}_{\beta\gamma}-\overline{\Gamma^{\alpha}_{\beta\gamma}}).$$

By calculating the curvature tensor directly from (2.6) under the Kaehlerian condition, we have the following essential components of $R^{A}{}_{BCD}$

(2. 7)
$$
\begin{aligned}
R^{\alpha}{}_{\beta\gamma\delta} &= \frac{1}{2}\left(K^{\alpha}{}_{\beta\gamma\bar{\delta}}+K^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}\delta}-K^{\alpha}{}_{\beta\delta\bar{\gamma}}-K^{\bar{\alpha}}{}_{\bar{\beta}\bar{\delta}\gamma}\right),\\
R^{\alpha}{}_{\beta\gamma\bar{\delta}} &= \frac{i}{2}\left(-K^{\alpha}{}_{\beta\gamma\bar{\delta}}+K^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}\delta}-K^{\alpha}{}_{\beta\delta\bar{\gamma}}+K^{\bar{\alpha}}{}_{\bar{\beta}\bar{\delta}\gamma}\right),\\
R^{\alpha}{}_{\bar{\beta}\gamma\delta} &= \frac{i}{2}\left(K^{\alpha}{}_{\beta\gamma\bar{\delta}}-K^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}\delta}-K^{\alpha}{}_{\beta\delta\bar{\gamma}}+K^{\bar{\alpha}}{}_{\bar{\beta}\bar{\delta}\gamma}\right),\\
R^{\alpha}{}_{\bar{\beta}\gamma\bar{\delta}} &= \frac{1}{2}\left(K^{\alpha}{}_{\beta\gamma\bar{\delta}}+K^{\bar{\alpha}}{}_{\bar{\beta}\bar{\gamma}\delta}+K^{\alpha}{}_{\beta\delta\bar{\gamma}}+K^{\bar{\alpha}}{}_{\bar{\beta}\bar{\delta}\gamma}\right),
\end{aligned}
$$

and the other components are related to (2.7) by

(2. 8)
$$
\begin{aligned}
R^{A}{}_{B\bar{\gamma}\delta} &= -R^{A}{}_{B\gamma\bar{\delta}}, & R^{A}{}_{B\bar{\gamma}\bar{\delta}} &= R^{A}{}_{B\gamma\delta},\\
R^{\bar{\alpha}}{}_{\beta CD} &= -R^{\alpha}{}_{\bar{\beta}CD}, & R^{\bar{\alpha}}{}_{\bar{\beta}CD} &= R^{\alpha}{}_{\beta CD},
\end{aligned}
$$

where $K^{\alpha}{}_{\beta\gamma\bar{\delta}}$ is the curvature tensor in the Kaehlerian manifold K_n i.e.

$$K^{\alpha}{}_{\beta\gamma\bar{\delta}} = \frac{\partial}{\partial \bar{z}^{\delta}}\Gamma^{\alpha}_{\beta\gamma} \qquad (conj.)$$

and $R^{A}{}_{BCD}$ was defined by

$$R^{A}{}_{BCD} = \frac{\partial}{\partial x^{D}}\left\{ {A \atop BC} \right\} - \frac{\partial}{\partial x^{C}}\left\{ {A \atop BD} \right\} + \left\{ {E \atop BC} \right\}\left\{ {A \atop ED} \right\} - \left\{ {E \atop BD} \right\}\left\{ {A \atop EC} \right\}.$$

From (2.7) we have

(2. 9)
$$R^{\alpha}{}_{\bar{\beta}\gamma\bar{\delta}} - R^{\alpha}{}_{\bar{\beta}\delta\bar{\gamma}} = 0, \qquad R^{\alpha}{}_{\beta\gamma\bar{\delta}} + R^{\alpha}{}_{\beta\bar{\gamma}\delta} = 0,$$

then, the Ricci tensr $R_{BC} = R^{A}{}_{BCA}$ is the form

(2.10)
$$R_{\beta\gamma} = R_{\bar{\beta}\bar{\gamma}}, \qquad R_{\beta\bar{\gamma}} = -R_{\bar{\beta}\gamma}$$

therefore, we have the following

THEOREM 2.3. *In the real Riemannian manifold X_{2n} whose metric is related by (1.2) to the metric of the Kaehlerian manifofd K_n, the matrix (R_{AB}) is same type as the matrix (b_{AB}) i.e.*

$$(R_{AB}) = \begin{pmatrix} S & T \\ -T & S \end{pmatrix}$$

where the $(n\times n)$ matrix $S=(R_{\alpha\beta})$ is symmetric, and the $(n\times n)$ matrix $T=(R_{\alpha\bar{\beta}})$ is skew-symmetric.

By (2.7), we have

$$
\begin{aligned}
&R_{\alpha\beta\gamma\delta}=K_{\alpha\bar{\beta}\bar{\gamma}\delta}-K_{\alpha\bar{\beta}\bar{\delta}\gamma}+K_{\bar{\alpha}\beta\gamma\bar{\delta}}-K_{\bar{\alpha}\beta\delta\bar{\gamma}},\\
(2.11)\qquad &R_{\alpha\beta\gamma\bar{\delta}}=i(K_{\alpha\bar{\beta}\bar{\gamma}\delta}+K_{\alpha\bar{\beta}\bar{\delta}\gamma}-K_{\bar{\alpha}\beta\gamma\bar{\delta}}-K_{\bar{\alpha}\beta\delta\bar{\gamma}}),\\
&R_{\alpha\bar{\beta}\gamma\delta}=i(-K_{\alpha\bar{\beta}\bar{\gamma}\delta}+K_{\alpha\bar{\beta}\bar{\delta}\gamma}+K_{\bar{\alpha}\beta\gamma\bar{\delta}}-K_{\bar{\alpha}\beta\delta\bar{\gamma}}),\\
&R_{\alpha\bar{\beta}\gamma\bar{\delta}}=K_{\alpha\bar{\beta}\bar{\gamma}\delta}+K_{\alpha\bar{\beta}\bar{\delta}\gamma}+K_{\bar{\alpha}\beta\gamma\bar{\delta}}+K_{\bar{\alpha}\beta\delta\bar{\gamma}},
\end{aligned}
$$

and the other components of R_{ABCD} are obtained by the following relations which are given by (2.8)

$$
(2.12)\qquad
\begin{aligned}
&R_{AB\bar{\gamma}\delta}=-R_{AB\gamma\bar{\delta}}, &\qquad R_{AB\bar{\gamma}\bar{\delta}}=R_{AB\gamma\delta},\\
&R_{\bar{\alpha}BCD}=-R_{\alpha\bar{B}CD}, &\qquad R_{\bar{\alpha}\bar{B}CD}=R_{\alpha BCD},
\end{aligned}
$$

then, the curvature tensor and the Ricci tensor in the Kaehlerian manifold K_n are the following forms

$$
(2.13)\qquad K^{\alpha}{}_{\beta\gamma\bar{\delta}}=\frac{1}{2}[(R^{\alpha}{}_{\beta\gamma\delta}+R^{\alpha}{}_{\bar{\beta}\gamma\bar{\delta}})+i(R^{\alpha}{}_{\beta\gamma\bar{\delta}}-R^{\alpha}{}_{\bar{\beta}\gamma\delta})],
$$

$$
(2.14)\qquad
\begin{aligned}
&K_{\bar{\alpha}\beta\gamma\bar{\delta}}=\frac{1}{4}[(R_{\alpha\beta\gamma\delta}-R_{\bar{\alpha}\beta\gamma\bar{\delta}})+i(R_{\bar{\alpha}\beta\gamma\delta}+R_{\alpha\beta\gamma\bar{\delta}})],\\
&K_{\alpha\bar{\beta}\gamma\bar{\delta}}=\frac{1}{4}[(R_{\alpha\beta\gamma\delta}-R_{\alpha\bar{\beta}\gamma\bar{\delta}})+i(R_{\alpha\bar{\beta}\gamma\delta}+R_{\alpha\beta\gamma\bar{\delta}})],
\end{aligned}
$$

$$
(2.15)\qquad K_{\beta\bar{\gamma}}=\frac{1}{2}(R_{\beta\gamma}+i\,R_{\beta\bar{\gamma}}).
$$

3. Harmonic and Killing vectors

For a contravariant vector η^A in the real Riemannian manifold X_{2n}, we can put ξ^A, as a contravariant vector in the Kaehlerian manifold K_n by the following relations

$$
(3.\ 1)\qquad \eta^A=\frac{\partial x^A}{\partial z^B}\xi^B
$$

then, by (1.5) we get

$$
(3.\ 2)\qquad \xi^{\alpha}=\eta^{\alpha}+i\eta^{\bar{\alpha}},\qquad \xi^{\bar{\alpha}}=\eta^{\alpha}-i\eta^{\bar{\alpha}}
$$

By denoting "|" and "," as the covariant differentiations with respect to the tensor b_{AB}, and the Kaehlerian metric tensor g_{AB} respectively, we have

$$(3.3)\quad \begin{aligned} \eta^{\alpha}{}_{|\beta} &= \frac{1}{2}(\xi^{\alpha}{}_{,\beta}+\xi^{\alpha}{}_{,\bar\beta}+\xi^{\bar\alpha}{}_{,\beta}+\xi^{\bar\alpha}{}_{,\bar\beta}) \\ \eta^{\alpha}{}_{|\bar\beta} &= \frac{i}{2}(\xi^{\alpha}{}_{,\beta}-\xi^{\alpha}{}_{,\bar\beta}+\xi^{\bar\alpha}{}_{,\beta}-\xi^{\bar\alpha}{}_{,\bar\beta}) \\ \eta^{\bar\alpha}{}_{|\beta} &= \frac{i}{2}(-\xi^{\alpha}{}_{,\beta}-\xi^{\alpha}{}_{,\bar\beta}+\xi^{\bar\alpha}{}_{,\beta}+\xi^{\bar\alpha}{}_{,\bar\beta}) \\ \eta^{\bar\alpha}{}_{|\bar\beta} &= \frac{1}{2}(\xi^{\alpha}{}_{,\beta}-\xi^{\alpha}{}_{,\bar\beta}-\xi^{\bar\alpha}{}_{,\beta}+\xi^{\bar\alpha}{}_{,\bar\beta}) \end{aligned}$$

$$(3.4)\quad \eta^{A}{}_{|A}=\xi^{A}{}_{,A}$$

and furthermore the relations

$$(3,5)\quad b^{BC}\eta^{A}{}_{|B|C}-R^{A}{}_{B}\eta^{B}=0,$$

which are a necessary and sufficient condition that a vector η^{A} in a compact orientable X_{2n} be a harmonic one [2] [4], are equivalent to

$$(3.6)\quad g^{BC}\xi^{\alpha}{}_{,B,C}-g^{\alpha\bar\beta}K_{\bar\beta\gamma}\xi^{\gamma}=0. \qquad (conj.)$$

Similarly, the relations

$$(3.7)\quad b^{BC}\eta^{A}{}_{|B|C}+R^{A}{}_{B}\eta^{B}=0, \quad \eta^{A}{}_{|A}=0.$$

which are a condition that η^{A} be a Killing vector [2], [4], are equivalent to

$$(3.8)\quad g^{BC}\xi^{\alpha}{}_{,B,C}+g^{\alpha\bar\beta}K_{\bar\beta\gamma}\xi^{\gamma}=0, (conj.) \quad \xi^{A}{}_{,A}=0,$$

therefore, if a vactor η^{A} in a compact orientable X_{2n} whose metric is related by (1.2) to the metric of K_n, is a harmonic or Killing one, then a vector ξ^{A} in a compact K_n is also harmonic or Killing one respectively. and conversely. By putting

$$(3.9)\quad \eta_{A}=b_{AB}\eta^{B}, \qquad \xi_{A}=g_{AB}\xi^{B},$$

we have by (3.3)

$$(3.10)\quad \begin{aligned} \eta_{\alpha|\beta} &= \xi_{\alpha,\beta}+\xi_{\alpha,\bar\beta}+\xi_{\bar\alpha,\beta}+\xi_{\bar\alpha,\bar\beta}, \\ \eta_{\alpha|\bar\beta} &= i(\xi_{\alpha,\beta}-\xi_{\alpha,\bar\beta}+\xi_{\bar\alpha,\beta}-\xi_{\bar\alpha,\bar\beta}), \\ \eta_{\bar\alpha|\beta} &= i(\xi_{\alpha,\beta}+\xi_{\alpha,\bar\beta}-\xi_{\bar\alpha,\beta}-\xi_{\bar\alpha,\bar\beta}), \\ \eta_{\bar\alpha|\bar\beta} &= -\xi_{\alpha,\beta}+\xi_{\alpha,\bar\beta}+\xi_{\bar\alpha,\beta}-\xi_{\bar\alpha,\bar\beta}. \end{aligned}$$

on the other hand, if ξ_A in a compact K_n is a harmonic vector, then by theorem 8.13 of [2], all ξ_α are analytic in (z^α) and all $\xi_{\bar\alpha}$ are analytic in $(\bar z^\alpha)$. And if ξ^A is a Killing vector and $\xi^{\alpha}{}_{,\alpha}=0$, by theorem 8.19 of [2], all ξ^α are analytic in (z^α), and all $\xi^{\bar\alpha}$ are analytic in $(\bar z^\alpha)$. Therefore, by (3.3) and (3.10) we have the following

THEOREM 3.1. *If a vector ξ^A in a compact Kaehlerian manifold K_n is a Killing one and $\xi^\alpha{}_{,\alpha}=0$, then for the components η^A in a compact orientable Riemannian manifold X_{2n} whose metric is related by* (1.2) *to the Kaehlerian metric, the matrix $(\eta^A{}_{|B})$ is the following form*:

$$(\eta^A{}_{|B})=\begin{pmatrix} C & D \\ -D & C \end{pmatrix}$$

where $C=(\eta^\alpha{}_{|\beta})$, $D=(\eta^\alpha{}_{|\bar\beta})$.

Similarly, if ξ_A is a harmonic vector, the matrix $(\eta_{A|B})$ is the following form:

$$(\eta_{A|B})=\begin{pmatrix} E & F \\ F & -E \end{pmatrix}$$

where both $E=(\eta_{\alpha|\beta})$ and $F=(\eta_{\alpha|\bar\beta})$ are symmetric.

From (3.2) and (3.9), we obtain

$$\xi_\alpha=\frac{1}{2}(\eta_\alpha-i\eta_{\bar\alpha}),\quad \xi_{\bar\alpha}=\frac{1}{2}(\eta_\alpha+i\,\eta_{\bar\alpha}), \tag{3.11}$$

and a vector η^A is called a conformal Killing vector if

$$\mathscr{L}b_{AB}=\eta_{A|B}+\eta_{B|A}=2\,\phi\, b_{AB} \tag{3.12}$$

where ϕ is a scalar function.

By using the relations (2.2) and (3.10), the condition (3.12) is equivalent to

$$\xi_{\alpha,\beta}=0,\quad \xi_{\bar\alpha,\bar\beta}=0,\quad \xi_{\alpha,\bar\beta}=2\,\phi\, g_{\alpha\bar\beta},\quad \xi_{\bar\alpha,\beta}=2\,\phi\, g_{\bar\alpha\beta},$$

therefore we have the following

THEOREM 3.2. *A vector η^A in a compact orientable Riemannian manifold X_{2n} is a conformal Killing vector, if and only if the vector ξ^A (which is related to η^A with* (3.1)*) in a compact Kaehlerian manifold K_n, satisfies the relations*

$$\xi_{A,B}=2\,\phi\, g_{AB}.$$

Finally, we consider the form

$$\mathscr{L}\left\{ {A \atop BC} \right\}=\eta^A{}_{|B|C}+R^A{}_{BCD}\,\eta^D \tag{3.13}$$

then, by the relations (2.4), (2.7) and (3.3) we have

$$\xi^{\alpha}{}_{,\beta,\gamma}=0,\quad \xi^{\bar{\alpha}}{}_{,\beta,\gamma}=0,\quad \xi^{\alpha}{}_{,\beta,\bar{\gamma}}=0,\quad \xi^{\bar{\alpha}}{}_{,\beta,\bar{\gamma}}=0,$$

furthermore we have

$$\mathscr{L}\left\{ {\alpha \atop \beta\gamma} \right\}=\frac{1}{2}\left(L\Gamma^{\alpha}_{\beta\gamma}+\overline{L\Gamma^{\alpha}_{\beta\gamma}}\right),$$

(3.14)

$$\mathscr{L}\left\{ {\alpha \atop \beta\bar{\gamma}} \right\}=\frac{i}{2}\left(L\Gamma^{\alpha}_{\beta\gamma}-\overline{L\Gamma^{\alpha}_{\beta\gamma}}\right),$$

where

$$L\Gamma^{\alpha}_{\beta\gamma}=\xi^{\alpha}{}_{,\beta,\gamma}+K^{\alpha}{}_{\beta\gamma\delta}\,\xi^{\delta} \qquad (conj.) \tag{3.15}$$

then, the condition $\mathscr{L}\left\{ {A \atop BC} \right\}=0$ is equivalent to $L\Gamma^{\alpha}_{\beta\gamma}=0$. Therefore, if an infinitesimal point transformation

$$'x^{A}=x^{A}+\eta^{A}(x)\delta t$$

is an affine collineation in X_{2n}, then an infinitesimal point transformation

$$'z^{A}=z^{A}+\xi^{A}(z)\,\delta t$$

is also an affine collineation in K_n and conversely [2].

Oct. 1959

Mathematical Department
Liberal Arts and Science College
Kyungpook University

REFERENCES

[1] K. Yano and I. Mogi: *On real representations of Kaehlerian manifolds*, Ann. of Math., Vol.61, No.1 (1955).

[2] K. Yano and S. Bochner: *Curvature and Betti numbers*, Ann. of Math. Studies No. 32, Princeton (1953).

[3] William M. Boothby: *Hermitian manifolds with zero curvature*, Michigan Math. Journal Vol.5 No.2 (1958).

[4] K. Yano: *Some integral formulas and their applications*, Michigan Math. Journal Vol.5 No.1 (1958).