ON A QUASI-ORDERED GROUP

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Introduction

By a *quasi-order* on a set is meant a reflexive, transitive binary relation $\leqq$. In this paper we shall consider a quasi-ordered group in the following sense: A group Q is called a *quasi-ordered group* (=qo-group), when in Q is defined a quasi-order $a \leqq b$, preserved under the group operation $+$:

$$a \leqq b \text{ implies } a+c \leqq b+c \text{ and } c+a \leqq c+b \text{ for all } c \text{ in } Q.$$

This seems to be a natural generalization of the usual concept of partial ordered group (=po-group).

Simple examples are provided with the followings

EXAMPLE 1. Let Q be the additive group of all real functions defined on the unit square $0 \leqq x, y \leqq 1$. And let $f \geqq g$ meant that $f(x, y) \geqq g(x, y)$ except on a set of measure zero (or except on a non-void fixed proper subset of the unit square).

EXAMPLE 2. Let Q be the additive group of all (n, n)-matrices with real elements. And let $(a_{ik}) \geqq (b_{ik})$ meant that $a_{ii} \geqq b_{ii}$ for all i (diagonal elements).

This paper is divided into three sections. § 2 is concerned with definitions and the properties of qo-group which are slightly generalized theorem of po-groups and l-groups. And § 3 deals with completeness of quasi-ordered set with l.u.b. and g.l.b. of the subset in its interval topology, which is analogy to Frink's theorem on a lattice. And finally, we shall prove qo-group is homogeneous in its interval topology.

2. Definitions and qo-groups Let Q be a quasi-ordered set under binary relation $\leqq$, and A be a subset of Q.

We write, [2]

$$M(A)=\{y\in Q : x\leqq y \text{ for some } x\in A\}.$$
$$L(A)=\{y\in Q : y\leqq x \text{ for some } x\in A\}.$$
$$E(A)=L(A)\cap M(A)$$

LEMMA 1 *If Q is a qo-group, and A is a subgroup (normal subgroup) of Q, then E(A) is also a subgroup (normal subgroup).*

The proof of Lemma 1 is trivial. And we omit the details.
Therefore, $E(o)$ is a normal subgroup of Q, (where 0 is an identity of Q) which is called *the kernel* of Q.

E. Schröder has proved that the relation $x\equiv y$ which means $x\leqq y$ and $y\leqq x$ in a quasi-ordered set is an equivalent relation, and if equivalent elements are identified, $\leqq$ becomes a partial ordering. [1]

We can easily see that in a qo-group, x, y are equivalent elements if and only if $x-y\in E(o)$. Hence if all elements of the congruence class to which 0 belongs are identified with 0, the quasi-ordering $\leqq$ becomes a partial ordering. Thus we can partially order factor gorup $Q/E(o)$ by defining $E(a)(=a+E(o))<E(b)$ if $E(a)\neq E(b)$ and $a<b$ (i. e. $a\notin E(b)$), $Q/E(o)$ is then a po-group, and the canonical mapping $a\rightarrow E(a)$ is a homomorphism of Q onto $Q/E(o)$ as a groups and a quasi-ordered sets. Hence we have the following theorem

THEOREM 1 *Let Q be a qo-group, then the factor group $Q/E(o)$ is a po-group. And the canonical mapping $a\rightarrow E(a)$ is a homomorphism of Q onto $Q/E(o)$.*

COROLLARY *A qo-group Q is a po-group if and only if the kernel of Q be a single subset of zero of Q.*

We now extend the theorems of po-groups and l-groups to the case of quasi-ordering.

THEOREM 2 *Let Q be a qo-group with the kernel N, and Q^+ the set of all positive elements : $a\geqq 0$, in Q. Q^+ has then following properties:*

(i) *$x\in Q^+$ and $-x\in Q^+$ if and only if $x\in N$*

(ii) *If* x, $y \in Q^+$, *then* $x+y \in Q^+$
(iii) *For all* $x \in Q^+$ $x+Q^+=Q^++x$.

Conversely, if a group Q contains a normal subgroup N and a subset Q^+ *of* Q *having the properties* (i), (ii) *and* (iii). *We can then introduce a quasi-order in* Q *so that it becomes a qo-group with the kernel* N, *defining* $x \geqq y$ *if* $x-y \in Q^+$ (*or* $-y+x \in Q^+$). (*cf.*[1] Theorem 1. *P.* 214)

PROOF The former part is almost obvious.

We prove the converse. From (i) we get $o \in N \subset Q^+$, Thus we have the reflexive. Again, from (ii) and (iii) we have the transitivity and $a \leqq b$ implies $a+c \leqq b+c$ *and* $c+a \leqq c+b$. Finally, we have to note that the normal subgsoup N is the kernel of Q : For, $x \in N$, we get $x \in Q^+$, $-x \in Q^+$ from (i). Thus $x \in E(o)$ of Q. Conversely, $x \in E(o)$ implies $x \in Q^+$, $-x \in Q^+$, Thus, by (i) $x \in N$.

L. E. Ward [2] has defined that the concept of g.l.b. and l.u.b. of a quasi-ordered set, which seems to be a suitable generalization of the usual concept of complete lattice, as following :

Let y be an element of quasi-ordered set X is ***minimal*** (***maximal***), whenever $x \leqq y$ ($y \leqq x$) in X implies $y \leqq x$ ($x \leqq y$). And let A be a subset of quasi-ordered set X, The element $x \in X$ is *an upper* (*lower*) *bound* for A provided $a \leqq x$ ($x \leqq a$) for all $a \in A$. The element x is *a least upper* (*greatest lower*) *bound* for A if x is a minimal (maximal) element of the set of upper (lower) bounds of A, dnote by $\bigvee$A ($\bigwedge$A). The quasi ordered set is $\bigvee\bigwedge$ *quasi-odered* ($\bigvee\bigwedge$*complete quasi-ordered*) if it has both a g.l.b. and a l.u.b. of any two elements (of any subset). We denote by $\bigvee\bigwedge$qos ($\bigvee\bigwedge$cqos). We have to note that the operations $\bigwedge$A, $\bigvee$A are not always unique operations in this case.

LEMMA 2 *Let Q be a* $\bigvee\bigwedge$*qo-group with the kernel* N, *Then the relation* $x \equiv y$ *which means* $x-y \in N$ *is a congruence relation* : $x \equiv a$ *and* $y \equiv b$ implies $x \bigvee y \equiv a \bigvee b$ *and* $x \bigwedge y \equiv a \bigwedge b$.

PROOF $x \equiv a$ and $y \equiv b$ imply $x-a$, $y-b \in E(o)(=N)$, Thus $a \leqq x$, $x \leqq a$, $y \leqq b$ and $b \leqq y$. And we have $L(x)=L(a)$ and $L(y)=L(b)$.

Therefore, $L(x)\cap L(y)=L(a)\cap L(b)$. Since $x\vee y$ is a maximal element of $L(x)\cap L(y)$ and $a\vee b$ is also a maximal element of $L(a)\cap L(b)$, we have $x\wedge y\equiv a\wedge b$. And dually.

LEMMA 3 *A qo-group Q with the kernel N is* a $\vee\wedge$*qo-group if and only if for all* $a\in Q$, *all* $x\in N$, $a\vee x$ *exists*

PROOF If Q is an $\wedge\vee$ qo-group, then obviously $a\vee x$ exists for all $a\in Q$, all $x\in N$. Conversely, let $a, b\in Q$. By the hypothesis, there exists the element $z\equiv o\vee(b-a)$ in Q. And we have $z+a\equiv a\vee b$. In fact, $z\equiv o\vee(b-a)$ implies that $z+a$ is an upper bound of a, b. If $w\in M(a)\cap M(b)$, then we see easily that $w-a\in M(o)\cap M(b-a)$. If $w\leqq z+a$ for some $w\in M(a)\cap M(b)$, since z is a mimimal element of $M(o)\cap M(b-a)$ we have $z+a\leqq w$. Thus the element $z+a$ is a minimal element of $M(a)\cap M(b)$. On the other hand we see that $-(-a\vee -b)\equiv a\wedge b$. The proof is complete.

Hence we have the following theorem

THEOREM 3 *Let Q be a qo-group with the kermel N, and if for all* $a\in Q$, *all* $x\in N$ $a\vee x$ *exists. Then the factor group* Q/N *is an l-group.*

COROLLARY *In any* $\wedge\vee$*qo-group with the kernel N, we have the following statements* [1]

(i) $a-(a\wedge b)+b\equiv b\vee a$ (ii) $a+b\equiv(a\vee b)+(a\wedge b)$

(iii) $a\wedge(b\vee c)\equiv(a\wedge b)\vee(a\wedge c)$

(iv) $a\wedge b$, $a\wedge c\in N$ *implies* $a\wedge(b+c)\in N$ *and dual.*

3 Interval topology in a quasi-ordered set

L. E. Ward [2] has also suggested that it is possible to introdeuce an interval topology for quasi-ordered sets. It is that topology which has for a subbase for closed sets all sets of the form $L(x)$ or $M(x)$, where x is a member of the quasi-ordered set.

We now extend the result of Frink to the case of $\wedge\vee$qo-set.
Let the closed interval $[a, b]$ be set of all elements x with $a\leqq x\leqq b$.

Then, if $a \equiv x$, $b \equiv y$ we have $[a, b]=[x, y]$.

THEOREM 4 *An* $\wedge\vee$*qos* Q *is a* $\wedge\vee$*cqos if and only if* Q *is compact in its interval topology.*

PROOF The necessity is easily given by the same manner of Frink. Conversely, let S be a non-null subset of Q. By the hypothesis and ([2] Theorem 1), Q has a minimal element. And each $L(x)$ being compact, $\cap\{L(x) : x \in S\}$ is non-null compact set. Therefore, $\cap\{L(x) : x \in S\}$ has a maximal element, which is a g.l.b. of S. And dually.

LEMMA 4 *Let* Q *be a qo-set and* $a \in Q$. *Then* $E(a)$ *is the closure of single set* $\{a\}$ *i.e.* $E(a)=\bar{a}$ *in its interval topology.*

PROOF Let C be a closed set including $\{a\}$. We assert that $E(a) \subset C$. For, if $a \in L(x)$ $(M(x))$, then $E(a) \subset L(x)$ $(M(x))$ by the transitivity of the qo-set. Since $E(a)=M(a) \cap L(a)$, $E(a)$ is a closed set including $\{a\}$. Hence we have $E(a)=\bar{a}$.

Therefore, we have proved

THEOREM 5 *A qo-sed* Q *is* T_1*-space in its interval topology if and only if* Q *is a poset.*

COROLLARY *A qo-group is* T_1*-space in its interval topology if and only if the kernel of the qo-group is the single set of zero of the qo-group.*

E.S. Northam [4] has proved that an l-group need not always be a topological group. Therefore, it is also true that qo-group is not always a topological group. Finally, we investigate the homogeneity of qo-group in its interval topology. (Q is *homogeneous* if for any two elements p and q of Q, there exists a topological transformation of Q into itself which transform p into q)

LEMMA 5 *Let* Q *be a qo-group. Suppose the mapping* f: $x \to a+x$, *where* a *is a fixed element of* Q. *Then* f *is a topological mapping of the space* Q *into itself in its interval topology.*

PROOF It is obvious that f is one-to-one. Furthermore, f is continuous. For, if S be a closed subset of Q, then S is the finite union of the intersection of arbitrary number of closed intervals $L(x)$, $M(x)$, $(x \in Q)$. We see easily that

$$a+L(x)=L(a+x),\ a+M(x)=M(a+x) \text{ for all } a,\ x \in Q.$$

Hence $f^{-1}(S)=-a+S$ is also the finite union of the intersection of arbitrary number of closed intervals $L(-a+x)$, $M(-a+x)$. Thus $f^{-1}(S)$ is a closed set, which the proof is complete.

COROLLARY *Let F be a closed subset, U an open set, P an arbitrary set and x some element of a qo-group. Then $x+F$ is a closed set, while $P+U$ is an open set.*

THEOREM 6 *A qo-group is homogeneous in its interval topology.*

Hence, from the homogeneity, it follows that it is sufficient to state and verify its local properties for a single element only.

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