ON THE PARAMETER GROUP MANIFOLDS

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Introduction.

In 1953 Harley Flanders [1] introduced the extended exterior differentiation on the differentiable manifold. The purpose of this paper is to study on the parameter group manifold, given some connections by N. Horie, [2] in the sense of the operation. In section 1, we shall define a linear space at any point of the group manifold and, from this, introduce the operation. In section 2 and 3, we shall introduce the connections, investigate the characteristic properties of their manifolds and compute some identities.

1. Basic definitions.

We consider the first parameter group $\mathcal{G}^{(+)}$ and the second parameter group $\mathcal{G}^{(-)}$ of continuous transformation group $\mathcal{G}$ of dimension r, whose equations may be represented by

$$a_3^\alpha = \varphi^\alpha(a_1, a_2), \quad (\alpha = 1, \cdots, r),$$

where a's are r essential parameter under the equations of the continuous transformation group $\mathcal{G}$, taking x's as n independent variables,

$$x'^i = f^i(x, a), \quad (i = 1, \cdots, n).$$

Let

$$\frac{\partial a_3^\alpha}{\partial a_2^\beta} = A_b^\alpha(a_3) A_\beta^b(a_2)^{*)},$$

$$\frac{\partial a_3^\alpha}{\partial a_1^\beta} = \bar{A}_b^\alpha(a_3) \bar{A}_\beta^b(a_1)$$

be the fundamental equations of $\mathcal{G}^{(+)}$ and $\mathcal{G}^{(-)}$ respectively, where the determinant $|A_b^\alpha|$ is not zero and $\|A_\alpha^b\|$ is the inverse one of $\|A_b^\alpha\|$, and $\bar{A}_b^\alpha$, $\bar{A}_\alpha^b$ are the ones of $\mathcal{G}^{(-)}$ corresponding to A_b^α, A_α^b of $\mathcal{G}^{(+)}$ res-

(*) We agree to sum over the possible values of the indices with respect to each index which uppears twice.

pectively. And we can consider the group manifolds $\mathscr{G}^{(+)}$, $\mathscr{G}^{(-)}$ as the differentiable manifolds, and assume that their classes are all C^{∞}.

Let A_{α} $(\alpha=1, \cdots, r)$ denote the pairs $(A^1_{\alpha}, A^2_{\alpha}, \cdots, A^r_{\alpha})$, then, since $|A^b_{\alpha}|$ is not zero, $A_1, \cdots, A_r$ form a basis of a vector space $\mathfrak{A}_P$ $(P \in \mathscr{G}^{(+)})$ i. e., $\mathfrak{A}_p=E\{A| A=C^{\alpha}A_{\alpha}\}$, where, C^{α} are constants. We call $\mathfrak{A}_p$ *the fundamental* 1-*vector space at P*. Let the dual space of $\mathfrak{A}_p$ denotes by $\mathscr{D}_p$, and the dual basis of $A_1, \cdots, A_r$ in $\mathfrak{A}_p$ by $D^1, \cdots, D^r$, then $\mathscr{D}_p=E\{D| D=C_{\alpha}D^{\alpha}\}$ forms a linear space of dimension r, called it *fundamental* 1-*form space at P*, where C_{α} are constants.

If A is any 1-vector at P, then $A=C^{\alpha}A_{\alpha}$, with constants C^{α}. A mapping ϕ which maps each point $P \in \mathscr{G}^{(+)}$ into a fundamental 1-vector $\phi(P)$ in $\mathfrak{A}_p$ is called a *fundamental* 1-*vector field* provided that in each local coordinate neighborhood with coordinate system $(a^1, \cdots, a^r)$ $\phi(P)$ is expressed by

$$\phi(P)=C^{\alpha}(a_1, \cdots, a_n)A_{\alpha}(a),$$

where $C^{\alpha}(a)$ are functions of classC^{∞}. In similar manner we define a *fundamental* 1-*form field*. When there is no danger of confusion, we shall refer to such an 1-vector field and an 1-form field simply as an 1-vector and an 1-form respectively.

Let $R=R(\mathscr{G})$ denote the ring of all C_{∞} functions on $\mathscr{G}$. And over the vector space $\mathfrak{A}_p$ at $P \in \mathscr{G}^{(+)}$, we may form the space $\wedge^p \mathscr{G}_p$ of *p-vectors at P* and the space $\wedge^q \mathscr{D}_p$ of *q-forms at P*, and in the same manner, we may define the *p-vector field* and *q-form field*. Furthermore, If the ring R acts both, then we have the linear spaces $\mathfrak{A}^p$ and $\mathscr{D}_q$ corresponding to $\wedge^p \mathfrak{A}_p$ and $\wedge^q \mathscr{D}_p$, where P represents each point in $\mathscr{G}^{(+)}$, respectively. Now, we consider the tensor products

$$\mathscr{T}^p_q=\mathscr{D}_q \otimes \mathfrak{A}^p.$$

At any rate, we act the ring R as a coefficient ring also, thus $\mathscr{T}^p_q$ forms a linear space. And, setting $\mathfrak{A}=\sum\oplus \mathfrak{A}^p$, $\mathscr{D}=\sum\oplus\mathscr{D}_q$, then each of these is an algebra over R, where multiplication is the Grassmann product. i. e., passing to homogeneous components, the operation on $\mathscr{T}^p_q \times \mathscr{T}^{p'}_{q'}$ to $\mathscr{T}^{p+p'}_{q+q'}$ given by linearity and

$$(DA)(D'A')=(DD')(AA'),\ D\in\mathscr{D}_q,\ D'\in\mathscr{D}_{q'},\ A\in\mathfrak{A}^p,\ A'\in\mathfrak{A}^{p'}.$$

This operation is associative and distributive, and on commutation it obeys the following:

$$MN=(-1)^{pp'+qq'}NM,\qquad M\in\mathscr{T}^p_q,\qquad N\in\mathscr{T}^{p'}_{q'}.$$

Next, we refer the *definition of extended exterior differentiation*. It is an operation d on each space $\mathscr{T}^p_q$ to $\mathscr{T}^p_{q+1}$ satisfying the followings:

(1) $d(M+N)=dM+dN,\quad M,\ N\in\mathscr{T}^p_q.$

(2) $d(MN)=dMN+(-1)^q MdN,\quad M\in\mathscr{T}^p_q,\ N\in\mathscr{T}^r_s$

(3) *d coincides with the affine connection on* $\mathscr{T}^1_0$, and *d coincides with the exterior differentiation on* $\mathscr{T}^0_q=\mathscr{D}_q$.

The existence and uniqueness of this operation is proved by Harley Flanders.

2. Curvature and torsion of $\mathscr{G}^{(+)}$.

From the above section, we have seen that $A_1,\cdots,A_r$ form a basis of the fundamental 1-vector space at a point $P\in\mathscr{G}^{(+)}$. By the definition of vector field, we may regard that $A_1(P),\cdots,\ A_r(P)$ are vector fields called the *fundamental frame*. If $A_1,\cdots,\ A_r$ is the frame with dual basis of forms $D^1,\cdots,\ D^n$ using of matrix notation, we shall set

$$(2.1)\qquad A=\begin{pmatrix}A_1\\ \vdots\\ A_r\end{pmatrix},\qquad D=(D^1,\cdots,\ D^r),$$

and it is convenient to us that a $(1,1)$ matrix is single element.

We set

$$(2.2)\qquad dP\equiv D^1A_1+\cdots\cdots+D^rA_r=DA,$$

and we call it the *displacement vector of* $\mathscr{G}^{(+)}$ ane $\mathscr{G}^{(0)}$ which we define in secion 3, and thus $dP\in\mathscr{T}^1_1$. Henceforth we assume that $\mathscr{G}^{(+)}$ is a manifold with an affine connection d. Moreover, the extended exterior differentiation d may be acted on the displacement vector dP, and we call $d^2P=d(dP)\in\mathscr{T}^1_2$ *torsion vector of* $\mathscr{G}^{(+)}$.

Since each $A_\alpha\in\mathscr{T}^1_0$, we have $dA_\alpha\in\mathscr{T}^1_1$ and thus there are unique 1-forms ω^α_β such that $dA_\alpha=\omega^\beta_\alpha A_\beta$, i. e., in matrix form,

(2. 3) $$dA=\Omega A, \textit{ where } \Omega=\|\omega_\beta^\alpha\|.$$

Taking ω_α^β so that $\omega_\alpha^\beta=L_{\alpha\tau}^\beta D^\tau=A_b^\beta \frac{\partial A_\alpha^b}{\partial a^\tau}D^\tau$, where $L_{\alpha\tau}^\beta\in R$, then we have $dA_\alpha=A_b^\beta \frac{\partial A_\alpha^b}{\partial a^\tau}D^\tau A_\beta$. Since $dA_\alpha=(dA_\alpha^1,\cdots,dA_\alpha^r)$, $A_\alpha^b\in R$, it follows that $dA_\alpha^a=\frac{\partial A_\alpha^a}{\partial a^\tau}D^\tau$, and thus we have

(2. 4) $$dA_\alpha=\frac{\partial A_\alpha}{\partial a^\tau}D^\tau.$$

This is converse, and hence we may regard that (2. 3) *and* (2. 4) *are equivalent*. Furthermore, if we put $D^\tau=da^\tau$, then we have $dA_\alpha=\frac{\partial A_\alpha}{\partial a^\tau}da^\tau$, and thus, in this case, we may regard that *this operation d is ordinary differentiation.* We call the 1-forms ω_α^β *the* (+)*-connection forms.*

If we now differentiate (2. 2) we obtain

$$d^2P=dDA-DdA=(dD-D\Omega)A.$$

Let us put $T=dD-D\Omega$, then we have

(2. 5) $$d^2P=TA, \qquad T=dD-D\Omega.$$

Thus T is (1, n) matrix of 2-forms which we shall call the *torsion form.*

In this section, we assume that $D^\alpha=da^\alpha$. Then, since $dD=d(da)=0$, we have

$$d^2P=-D\Omega A=-da^\alpha\,\omega_\alpha^\beta A_\beta=-da^\alpha\,(L_{\alpha\tau}^\beta da^\tau)A_\beta$$
$$=-\frac{1}{2}(L_{\alpha\tau}^\beta-L_{\tau\alpha}^\beta)\,da^\alpha da^\tau A_\beta.$$

Since $L_{\alpha\tau}^\beta-L_{\tau\alpha}^\beta=C_{ab}^e\,A_\alpha^a\,B_\tau^b\,A_e^\beta$, it reduces to

$$d^2P=-\frac{1}{2}C_{ab}^e A_\alpha^a\,A_\tau^b\,A_e^\beta\,da^\alpha\,da^\tau A_\beta.$$

If we put

(2. 6) $$d^2P=-T_{\beta\gamma}^\alpha da^\beta da^\gamma A_\alpha,$$

we have

(2. 7) $$T_{\beta\gamma}^\alpha=\frac{1}{2}C_{ab}^e A_\beta^a\,A_\gamma^b\,A_e^\alpha,$$

and thus the coefficients of torsion forms are represented by

$\frac{1}{2} C^e_{ab} A^a_\beta A^b_\gamma A^\alpha_e$.

If we differentiate the second of (2. 5), we have

$$dT + T\Omega = D\Theta, \qquad \Theta = d\Omega - \Omega^2. \tag{2. 8}$$

The (n, n)matrix Θ of 2-forms θ^β_α is called the *curvature matrix* and its elements are the *curvature forms of* $\mathscr{G}^{(+)}$. If we now compute the curvature form θ^β_α, then we have

$$\theta^\beta_\alpha = d\omega^\beta_\alpha - \omega^\beta_\tau \omega^\tau_\alpha = \frac{1}{2} L^\beta{}_{\alpha\lambda\mu}\, da^\lambda\, da^\mu,$$

where

$$L^\beta{}_{\alpha\lambda\mu} = \frac{\partial}{\partial a^\lambda} L^\beta_{\alpha\mu} - \frac{\partial}{\partial a^\mu} L^\beta_{\alpha\lambda} + L^\tau_{\alpha\mu} L^\beta_{\tau\lambda} - L^\tau_{\alpha\lambda} L^\beta_{\tau\mu},$$

and thus, since $L^\beta{}_{\alpha\lambda\mu} = 0$, we have $\theta^\beta_\alpha = 0$, i. e., Θ is zero matrix. Hence, (2. 8) are reduced to

$$dT = -T\Omega, \qquad d\Omega = \Omega^2. \tag{2, 9}$$

Writting the first of (2. 9) by the element form, it is $dT^\alpha = -T^\tau \omega^\alpha_\tau$. Let us compute this equation. Substituting $T^\alpha = T^\alpha_{\beta\gamma} da^\beta\, da^\gamma$ and $\omega^\alpha_\tau = L^\alpha_{\tau\sigma} da^\sigma$ into it, then we have

$$\left(\frac{\partial}{\partial a^{(\sigma}} T^\alpha_{\beta\gamma)} + L^\alpha_{\tau(\sigma} T^\tau_{\beta\gamma)}\right) da^\beta da^\gamma da^\sigma = 0. \quad (*)$$

Using of (2. 6) and $L^\alpha_{\beta\gamma} = A^\alpha_b \dfrac{\partial A^b_\beta}{\partial a^\gamma} = -A^b_\beta \dfrac{\partial A^\alpha_b}{\partial a^\gamma}$, then it reduce to

$$(C^e_{af} C^f_{bc} + C^e_{bf} C^f_{ca} + C^e_{cf} C^f_{ab})\, A^\alpha_e A^a_\sigma A^b_\beta A^c_\gamma\, da^\sigma da^\beta da^\gamma = 0.$$

Since the part of bracket is zero identically and this computation is conversible, we can see that the first of (2. 9) is satisfied identically.

We can compute the following identities:

$$d\Omega^{2k} = 0, \qquad d\Omega^{2k+1} = \Omega^{2k+2}, \quad (k \geqq 0) \tag{2.10}$$

$$d^3 P = 0, \tag{2.11}$$

(*) $K(\alpha\beta\gamma)$ denotes the sum of cyclic parts for indices α, β and γ.

(2.12) $d^2A=0,$

(2.13) $d^2T=0,$

For, (2.10) holds by induction, (2.11) from $d^3P=d(d^2P)$ $=d(-da\Omega A)=da(d\Omega A-\Omega dA)=0$ by virtue of (2.10), (2.12) from $d^2A=d(dA)=d(\Omega A)=0$ by (2.10), (2.13) from $d^2T=d(dT)$ $=d(-T\Omega)=-dT\Omega-Td\Omega=T\Omega^2-T\Omega^2=0.$

3. Curvature and torsion of $\mathcal{G}^{(0)}$.

Let us set the connection forms ω_α^β on (2. 3) so that

(3. 1) $$\omega_\alpha^\beta=\Gamma_{\alpha\tau}^\beta D^\tau, \text{ where } \Gamma_{\alpha\tau}^\beta=\frac{1}{2}A_b^\beta\left(\frac{\partial A_\alpha^b}{\partial a^\tau}+\frac{\partial A_\tau^b}{\partial a^\alpha}\right).$$

We denote this group manifold with connections (3. 1) by $\mathcal{G}^{(0)}$ and call this form (0)-*connection*. In $\mathcal{G}^{(0)}$, if we take D^τ as da^τ, since it holds (2. 3), we have

$$2dA_\alpha=A_b^\beta\left(\frac{\partial A_\alpha^b}{\partial a^\tau}+\frac{\partial A_\tau^b}{\partial a^\alpha}\right)D^\tau A_\beta,$$

and consequently,

$$2dA_\alpha^a=\frac{\partial A_\alpha^a}{\partial a^\tau}da^\tau+\frac{\partial A_\tau^a}{\partial a^\alpha}da^\tau$$

Exterior multiplying it by da^α and summing for α, then we obtain

$$da^\alpha\, dA_\alpha=0,$$

and thus,

(3. 2) $da\, dA=0.$

Hence, from (2. 3) and (3. 2), *on* $\mathcal{G}^{(0)}$ *we may hold*

(3. 3) $da\,\Omega A=0.$

Extended exterior differentiating (2. 2) and using of (3. 3), we have

$$d^2P=0$$

and hence, *the* $\mathcal{G}^{(0)}$ *the torsion forms* T *are vanish.*

Since the curvature matrix Θ are represented by the second of (2. 8), we may see that the curvature forms θ_α^β are computable to

$$\theta_\alpha^\beta=\frac{1}{2}\Gamma^\beta{}_{\alpha\lambda\mu}\, da^\lambda\, da^\mu,$$

where

$$\Gamma^{\beta}{}_{\alpha\lambda\mu}=\frac{\partial}{\partial a^{\lambda}}\Gamma^{\beta}_{\alpha\mu}-\frac{\partial}{\partial a^{\mu}}\Gamma^{\beta}_{\alpha\lambda}+\Gamma^{\tau}_{\alpha\mu}\Gamma^{\beta}_{\tau\lambda}-\Gamma^{\tau}_{\alpha\lambda}\Gamma^{\beta}_{\tau\mu},$$

and consequently, since $\Gamma^{\beta}{}_{\alpha\lambda\mu}=\frac{1}{4}C^{e}_{dc}C^{a}_{eb}A^{\beta}_{a}A^{b}_{\alpha}A^{c}_{\lambda}A^{d}_{\mu}$, the curvature forms are given by

(3. 4) $$\theta^{\beta}_{\alpha}=\frac{1}{4}C^{e}_{dc}C^{a}_{eb}A^{\beta}_{a}A^{b}_{\alpha}A^{c}_{\lambda}A^{d}_{\mu}\,da^{\lambda}\,da^{\mu},$$

From (3. 3), *we can compute the following identities satisfying on* $\mathcal{G}^{(0)}$.

(3. 5) $$da\ \Theta^{r}A=0, \quad (r\geqq 1),$$

(3. 6) $$da\ \Omega\Theta^{r}A=0, \quad (r\geqq 0).$$

For, these are provable from (3. 3) by induction, using of (2. 8) and the Bianchi identity:

(3. 7) $$d\Theta^{r}=\Omega\Theta^{r}-\Theta^{r}\Omega, \quad (r\geqq 1).$$

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REFERENCES

[1] H. Flanders; *Developement of an extended exterior differential calculus.* Trans. Amer. Math. Soc. vol. 75 (1953) pp. 311—326.

[2] N. Horie; *On the group space of the continuous-transformation group with a Riemannian metric.* Memo. Coll. Scie. Univ. Kyoto vol. XXX, No. 1 (1956).