# AN APPROXIMAION TO THE SECOND BOUNDARY VALUE PROBLEM

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## 1. Introduction

Beno Eckmann gave physical interpretations *grad* and *div* to the cohomology and homology operators $\delta$ and $\partial$. He gave another proof of the existence and the uniqueness of electric current on a network with given resistances and electromotoric forces by homology theoretical method, although the original proof was given by H. Weyl by different method. Eckmann also treated the 1-st boundary value problem, [1].

The purpose of this paper is to give a formulation of the 2-nd boundary value problem in discrete case.

The well known 2-nd boundary value problem in classic analysis states as follow:

Let $D_1$ be a bounded domain with sufficiently smooth boundary $B_1$ in an Euclidean space. A function $g(b)$, $b \in B_1$, is given on $B_1$. To find a harmonic function $f(d)$, $d \in D_1$, which satisfies

$$\triangle f = div \cdot grad f = 0 \quad \text{on } D_1,$$

whose normal derivative $\frac{\partial f}{\partial n} = g$ on $B_1$.

This problem is solvable if and only if

$$\int_{B_1} g ds = 0$$

and the solution $f$ is unique except adding an arbitrary constant.

## 2. Notations

Let an arbitrary 1-dimensional complex constructed in $D_1$ be $K_1^*$, and we call the set of vertices, 0-dimensional simplices, of $K^*$.

$I = \{i \mid i = a \ vertex \ of \ K^*\}$ = the set of inner points, the set of edges, 1-dimensional simplices, of $K^*$.

$D$=the set of domainal edges.

From several vertices which are close to the boundary $B_1$, we draw segments $C=\{c\}$ vertically to $B_1$, and denote the set of the other ends of $c$, $B=\{b\}$.

$C$=the set of connecting edges,

$B$=the set of boundary points.

We construct a complex $K$ from the edges of $C$, $D$ and the vertices of $I$, $B$. Then $K^*$ is a subcomplex of $K$. Note the following two properties:

i) Every vertex of any edge in $D$ belongs to $I$.

ii) Every simple closed path of $K$ consists of the edges of $D$.

These two properties can be expressed as

$$\mathscr{Z}_1 \subset V_D, \quad \partial V_D \subset V_I$$

(For the notations, see the following definitions.) In order to generalize this problem, we simply give the following definitions.

Complex $K$……a finite simplicial complex whose dimension is $n$.

$\mathscr{C}^p(K)$……the $p$-dimensional chain group of $K$ with real coefficients. This is an $n$-dimensional vector space. A chain is denoted by $C^p$.

The inner product of two chains $C^p$ and $C_1^p$……If

$$C^p=\sum g_i\sigma_i, \quad C_1^p=\sum g'_i\sigma_i$$

($\sigma_i$ are the $p$-dimensional simplices of $K$), then the inner product $C^p\cdot C_1^p$ is defined by $\sum g_i g'_i$.

$\partial$, $\delta$……the usual boundary and coboundary operations in $K$.

$\mathscr{Z}_p$, $Z^p$……the groups of all cycles and cocycles, respectively. These are linear subspaces of the vector space $\mathscr{C}^p$.

$\mathscr{B}_p$, $P^p$……the groups of all bounding cycles and cocycles respectively.

$D, C$……a partition of the set of all $(p+1)$-dimensional simplices of $K$.

$I$, $B$. ……a partition of the set of all $p$-dimensional simplices of $K$.

$V_D$, $V_C$, $V_I$, $V_B$……the vector spaces spanned by the simplices of $D$, $C$, $I$ and $B$.

Condition($E_p$) $\quad \mathscr{Z}_{p+1} \subset V_D, \ \partial V_D \subset V_I$.

$K^*$……$(p+1)$-dimensional complex made of the simplices of $D, I$ and their faces.

### 3. The 2-nd boundary value problem

Let $g(c)$ be a given function defined on $C$, i.e. $g$ is a vector of $V_C$. To find a $p$-dimensional chain (vector) $f$ such that

$$\begin{cases} \delta f = g & \text{on } C \\ \partial\delta f = 0 & \text{on } I. \end{cases}$$

THE MAIN THEOREM. *The above problem is solvable if and only if* $Z^p(K^*)\cdot\partial g=0$, *and the solution is unique modulo cocycles, provided that the conditicon* $(E_p)$ *holds.*

PROOF. Let us suppose that there exists at least one solution.
Then

$$\delta f = g + g_D \quad \text{for some } g_D \in V_D$$
$$\partial\delta f = \partial g + \partial g_D, \qquad \partial g = \partial\delta f - \partial g_D$$

Now,

$$Z^p(K^*)\cdot\partial\delta f \subset Z^p(K^*)\cdot V_B$$
$$\subset \mathscr{L}^p(K^*)\cdot V_B = V_I\cdot V_B = 0$$
$$Z^p(K^*)\cdot\partial g_D = Z^p(K^*)\cdot\partial_* g_D = \delta_* Z^p(K^*)\cdot g_D$$
$$= 0\cdot g_D = 0,$$

where $\partial_*$ and $\delta_*$ are the boundary and coboundary operations of $K^*$.
Hence

$$\partial g\cdot Z^p(K^*) = 0$$

Conversly let us suppose that the above relation holds.

$$\mathscr{L}^p = V_I + V_B = \mathscr{L}^p(K^*) + V_B = Z^p(K^*) + \mathscr{R}_p(K^*) + V_B.$$

Therefore

$$\partial g \in R_p(K^*) + V_B.$$

So we can find $g_D \in V_D$, $f_B \in V_B$ such that

$$\partial g = \partial g_D + f_B$$

Now let $\mathscr{P}_{p+1}$ be the orthognal projection of $g-g_D$ on $\mathscr{P}_{p+1}$, that is

$$(g-g_D-\mathscr{T}_{p+1})\cdot\mathscr{P}_{p+1}=0, \quad \mathscr{T}_{p+1}\in\mathscr{P}_{p+1},$$

then

$$g-g_D-\mathscr{T}_{p+1}\in P^{p+1},$$

and there exists an $f$ such that

$$\delta f = g - g_D - \mathscr{T}_{p+1}.$$

Because $\mathscr{T}_{p+1} \in \mathscr{J}_{p+1} \subset V_D,$

$$\delta f = g \quad \text{on } C.$$

And $\partial\delta f = \partial g - \partial g_D - \partial\mathscr{T}_{p+1} = \partial g - \partial g_D = f_B$

$$\partial\delta f = 0 \quad \text{on } I.$$

Finally, we will prove the uniqueness. Let $f_1$ be another solution of the problem,

$$\delta f = \delta f_1 = g \quad \text{on } C,$$
$$\partial\delta f = \partial\delta f_1 = 0 \quad \text{on } I.$$

Then

$$\delta(f - f_1) = 0 \quad \text{on } C, \text{ and}$$
$$h = \delta(f - f_1) \in V_D \qquad (1)$$
$$\partial h = 0 \quad \text{on } I, \text{ so } \partial h \in V_B \qquad (2)$$

But from (1), $\partial h \in \partial V_D \subset V_I.$

This, together with (2) yields $\partial h = 0,$

that is $h \in \mathscr{J}_{p+1}$

But $h = \delta(f - f_1) \in P^{p+1},$ so

$$h \in \mathscr{J}_{p+1} \cap P^{p+1} = 0.$$

## 4. Case $p=0$

The condition $Z^p(K^*) \cdot \partial g = 0$ becomes $Z^0(K^*) \cdot \partial g = 0$

Let $K_i^*$ $(i = 1, 2, \cdots, l)$ be the components of $K^*$. If $\zeta_i^0$ is the fundamental cocycle of $K_i^*$, then the condition is expressed as

$$\zeta_i^0 \cdot \partial g = 0 \quad \text{for all } i.$$

This means that the sum of the values of $g$ on those edges issuing from $K_i^*$ is 0 for all $i$. This is a similar result to

$$\int_{B_1} g\,ds = 0 \quad \text{for the continuous case.}$$

1-dimensional complex satisfying $(E_1)$ can be obtained as follows.

Let $K^*$ be an arbitrary 1-dimensional finite simplicial complex whose components we denote by $K_i^*$ $(i = 1, 2, \cdots, l)$. The edges and vertices of $K^*$ are the simplices of $D$ and $I$. We enlarge $K^*_i$ by putting arbitrary trees on its vertices, obtaining enlarged components $K^{**}_i$, we can connect arbitrary components $K^{**}_i$ to other components $K^{**}_j$ by trees. But if $K^{**}_i$ is connected to $K^{**}_j$, and $K^{**}_j$ is connected to $K^{**}_t$, then connecting $K^{**}_i$ to $K^{**}_t$ is forbidden.

Finally we can augment thus obtained complex by adding arbitrary separate trees which shall make components by themselves in the finally obtained complex $K$. The edges and vertices of $K$ which do not belong to $K^*$ are the vertices of $C$ and $B$.

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## REFERENCE

[1] Beno Eckmann, *Harmonische Funktionen und Randwertaufgaben in einem Komplex*, Commentarii Mathematici Helvetici, Vol. 17, 1944. pp, 240—255.