ON GENERAL HARMONIC AND KILLING VECTORS IN METRIC MANIFOLDS WITH GENERAL SYMMETRIC AFFINE CONNECTION

By Jae Koo Ahn

1. Introduction

The properties of harmonic and Killing vector were studied in Riemannian manifold by many authors, and they were expanded in the metric manifold with torsion.

In this note we shall investigate the properties of harmonic and Killing vectors in general metric manifolds with symmetric affine connection, which are generalized from the formers. As the result of this study we shall get some different properties from them in the metric manifolds with torsion, and it will be more general than in Riemannian manifolds.

Let us consider the n-dimensional compact manifold V^n on which there is given a positive definite metric

$$ds^2=g_{jk}dx^j\,dx^k$$

and symmetric affine connection Γ^i_{jk}, which satisfies the equations

$$\Gamma'^a_{bc}=\frac{\partial x'^a}{\partial x^i}\left(\frac{\partial^2 x^i}{\partial x'^b\,\partial x'^c}+\frac{\partial x^j}{\partial x'^b}\,\frac{\partial x^k}{\partial x'^c}\,\Gamma^i_{jk}\right)$$

for the coordinate transformation. Then we have the relations

(1) $\qquad \Gamma^i_{jk}=\left\{{i \atop jk}\right\}+a^i_{jk}$

between the connection and the Christoffel symbol, where a^i_{jk} are the components of a symmetric tensor with respect to the subscripts. For this tensor a^i_{jk}, we denote as the followings:

(2) $\qquad g^{kl}a^i_{jk}=a^{i\cdot l}_{\cdot j},\;\; g^{kl}\,g^{jm}\,a^i_{jk}=a^{i\cdot lm},\;\; g^{jk}\,a^i_{jk}=a^i.$

If we denote the covariant differentiations with respect to Γ^i_{jk} and $\left\{{i \atop jk}\right\}$ by comma and semi-colon respectively, we have the followings for the vectors ξ_j, ξ^i;

(3) $\qquad \xi_{j,k}=\xi_{j;k}-\xi_i\,a^i_{jk},$

$\qquad\quad \xi^i{}_{,k}=\xi^i{}_{;k}+\xi^j\,a^i_{jk},$

and for the tensor T^i_{jk},

(4) $$T^i_{jk,l}=T^i_{jk;l}+T^m_{jk}\,a^i_{ml}-T^i_{mk}\,a^m_{jl}-T^i_{jm}\,a^m_{kl}.$$

Accordingly, we aware that covariant differentiations of the fundamental tensors g_{jk} and g^{ij} with respect to Γ^i_{jk} are not zero, but they are represented by

$$g_{jk,l}=-g_{mk}\,a^m_{jl}-g_{jm}\,a^m_{kl},$$

$$g^{ij}{}_{,l}=g^{mj}\,a^i_{ml}+g^{im}\,a^j_{ml}.$$

2. The applications of Theorem of Bochner-Hopf.

Now, in a compact metric manifold with symmetric affine connection Γ^i_{jk} whose metric $ds^2=g_{jk}\,dx^j\,dx^k$ is positive definite, we take a scalar ϕ and set

$$\phi_{,k}=\frac{\partial\phi}{\partial x^k},$$

$$\phi_{,j,k}=\frac{\partial^2\phi}{\partial x^j\,\partial x^k}-\frac{\partial\phi}{\partial x^i}\,\Gamma^i_{jk},$$

Then we have

$$\Delta\phi\equiv g_{jk}\,\phi_{,j,k}=g^{jk}\frac{\partial^2\phi}{\partial x^j\partial x^k}-g^{jk}\frac{\partial\phi}{\partial x^i}\,\Gamma^i_{jk},$$

and hence, applying theorem of Bochner-Hopf. we obtain

LEMMA. *In a compact metric manifold with symmetric affine connection Γ^i_{jk}, if, for a scalar $\phi(x)$, we have*

$$\Delta\phi\equiv g^{jk}\,\phi_{,j,k}\geqq 0$$

n everywhere, then we get

$$\Delta\phi=0.$$

For a vector field $\xi_i(x)$, if we put

$$\phi(x)=\xi^i\,\xi_i=g^{ij}\,\xi_i\,\xi_j,$$

then we have

(5) $$\Delta\phi=2g^{jk}[g^{mn}(\xi_m\,\xi_{n,j}+\xi_m\,\xi_a\,a^a_{nj})]_{,k}$$

$$=2[g^{mn}g^{jk}\xi_m\xi_{n,j,k}+2(a^{m\cdot in}+a^{i\cdot mn})\xi_i\xi_{m,n}$$
$$+(a^{j\cdot m}_{\ \ n}a^{k\cdot n}_{\ \ m}+a^{m\cdot j}_{\ \ n}a^{k\cdot n}_{\ \ m}+g^{mn}g^{lj}a^k_{lm,n})\xi_j\xi_k$$
$$+g^{jk}g^{mn}\xi_{m,j}\xi_{n,k}]$$

On the other hand, we have

$$g^{jk}\xi_{m,j,k}-g^{jk}(\xi_{m,j}-\xi_{j,m})_{,k}-g^{jk}\xi_{j,k,m}=\xi_i\Gamma^i{}_m,$$

where $\Gamma^i{}_m=g^{jk}\Gamma^i_{jkm}$. If the vector ξ_i satisfies

$$(6)\qquad g^{jk}(\xi_{m,j}-\xi_{j,m})_{,k}+g^{jk}\xi_{j,k,m}=-g^{lj}a^k_{lm}\xi_{j,k},$$

then it is reduced to

$$(7)\qquad g^{jk}\xi_{m,j,k}=\xi_i\Gamma^i{}_m-2g^{lj}a^k_{lm}\xi_{j,k},$$

and consequently, using of Lemma, we have

THEOREM 1. *In a compact metric manifold with symmetric affine connection Γ^i_{jk}, if there exists non-zero vector ξ_i satisfying* (6) *and*

$$(8)\qquad (\Gamma^{jk}+a^{j\cdot m}_{\ \ n}a^{k\cdot n}_{\ \ m}+a^{m\cdot j}_{\ \ n}a^{k\cdot n}_{\ \ m}+g^{mn}g^{lk}a^j_{lm,n})\xi_j\xi_k$$
$$+2(a^{i\cdot mn}+a^{n\cdot ni}-a^{n\cdot mi})\xi_i\xi_{m,n}+g^{jm}g^{kn}\xi_{j,k}\xi_{m,n}\geqq 0,$$

then the equality holds.

Similarly we obtain

THEOREM 2. *In a compact metric manifold with symmetric affine connection, if there exists non-zero vector ξ_i satisfying*

$$(9)\qquad g^{jk}(\xi_{m,j}+\xi_{j,m})_{,k}-g^{jk}\xi_{j,k,m}=0$$

and

$$(10)\qquad (\Gamma^{jk}-a^{j\cdot m}_{\ \ n}a^{k\cdot n}_{\ \ m}-a^{m\cdot j}_{\ \ n}a^{k\cdot n}_{\ \ m}-g^{mn}g^{lj}a^k_{lm,n})\xi_j\xi_k$$
$$-2(a^{m\cdot ni}+a^{i\cdot mn})\xi_i\xi_{m,n}-g^{jm}g^{kn}\xi_{j,k}\xi_{m,n}\leqq 0,$$

then the equality holds.

3. General harmonic vector and Killing vector.

We shall call a vector ξ_i general harmonic vector, if it satisfies

$$(11)\qquad \xi_{i,j}=\xi_{j,i},\qquad g^{jk}\xi_{j,k}=0.$$

Such a vector satisfies evidently (6) and consequently (7), for $\phi=g^{ij}\xi_i\xi_j$, we have*

$$\Delta\phi=(\Gamma^{(jk)}+a^{(j\cdot|m|}{}_{n}a^{k)\cdot n}{}_{m}+a^{m\cdot(j}{}_{n}a^{k)\cdot n}{}_{m}+g^{mn}g^{l(j}a^{k)}_{lm,n})\,\xi_j\xi_k$$
$$+2a^{l\cdot(mn)}\xi_l\xi_{m,n}+(g^{jm}g^{kn}+g^{jn}g^{km})\,\xi_{j,k}\xi_{m,n},$$

thus we have

THEOREM 3. *In a compact metric manifold with symmetric affine connection, if the symmetric matrix*

$$(12)\qquad \begin{pmatrix} \Gamma^{(jk)}+a^{(j\cdot|m|}{}_{n}a^{k)\cdot n}{}_{m}+a^{m\cdot(j}{}_{n}a^{k)\cdot n}{}_{m}+g^{mn}g^{l(j}a^{k)}_{lm,n} & 2a^{l\cdot mn} \\ 2a^{l\cdot mn} & g^{jm}g^{kn}+g^{jn}g^{km} \end{pmatrix}$$

defines non-negative quadratic form in the variables ξ_l and $\xi_{mn}=\xi_{nm}$, then every general harmonic vector ξ_l must satisfy so that the quadratic form vanishes. If the matrix defines a positive definite form, then there is no general harmonic vector other than zero.

Here, when we put $a^{l\cdot mn}=0$, i. e., $a^{l}_{mn}=0$, then we have $\Gamma^{l}_{jk}=\{{}^{l}_{jk}\}$ and consequently,

$$\xi_{m,n}=\xi_{m;n}$$

and

the (1,1)*-element in matrix* $(12)=2R^{jk}$.

Moreover, if we put $R^{jk}=0$, then we have

$$\Delta\phi=2g^{jm}g^{kn}\xi_{j;k}\xi_{m;n}=0$$

for the general harmonic vector ξ_l, and consequently,

$$\xi_{m;n}=0,$$

or

$$\frac{\partial\xi_m}{\partial x^n}=\frac{\partial\xi_n}{\partial x^m},$$

and thus the general harmonic vector is gradient. Hence we have

THEOREM 4. *If the* (2,1)*-element in the matrix* (12) *vanishes, then our manifold is reduced to the compact Riemannian manifold,*

* $\Gamma^{(jk)}$ denotes by $\Gamma^{(jk)}=\Gamma^{jk}+\Gamma^{kj}$.

and furthermore, if the (1, 1)-elenent vanishes, then the general harmonic vector is gradient.

Next, we shall call a vector ξ_i general Killing vector, if it satisfies

(13) $\xi_{i,j}+\xi_{j,i}=0.$

For the general Killing vector, we have

$$g^{jk}\xi_{j,k}=0$$

and moreover, evidently,

$$g^{jk}\xi_{j,k,i}=0$$

Thus it satisfies (9) and is reduced to

$$g^{jk}\xi_{m,j,k}=-\xi_i\Gamma^{i}{}_{m}$$

Hence we have

THEOREM 5. *In a compact metric manifold with symmetric affine connection, if the matrix*

(14) $$\begin{pmatrix} \Gamma^{(jk)}-2a^{(j\cdot|m|}{}_{n}a^{k)\cdot n}{}_{m}-a^{m\cdot(j}{}_{n}a^{k)\cdot n}{}_{m}-g^{mn}g^{l(j}a^{k)}_{lm,n} & -a^{[s\cdot t]i} \\ -a^{[s\cdot t]i} & -(g^{jm}g^{kn}-g^{jn}g^{km}) \end{pmatrix}$$

defines a non-positive quadratic form in the variables ξ_i and $\xi_{mn}=-\xi_{nm}$, then every general Killing vector ξ_i must satisfy so that the quadratic form vanishes. If the matrix defines a positive definite form, then there is no general Killing vector other than zero.

4. Integration on our metric manifold.

In this section, our metric manifold with symmetric affine connection Γ^{i}_{jk} is compact and oriented, and suppose that the tensor a^{i}_{jk} satisfies the condition

(15) $a^{i}_{ji}=0.$

Then, for any vector v^i, we have

$$v^i{}_{,i}=v^i{}_{;i}=\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}}{\partial x^i}v^i,$$

By Theeorem of Green, we have

(16) $$\int v^{i}{}_{,i}\, dV=0,$$

the integral domain being the whole manifold, where dV is the volume element.

First, applying (16) to the vector $(g^{im}\xi_m)_{,j}\, g^{jn}\xi_n$ and using of Ricci identity, we obtain

$$\int[((\Gamma^{ij}+2a^{m\cdot i}_{\ \ n}a^{n\cdot j}_{\ \ m}+3a^{i\cdot m}_{\ \ n}a^{n\cdot j}_{\ \ m}+a^{i\cdot m}_{\ \ n}a^{j\cdot n}_{\ \ m}+a^{m}a^{i\cdot j}_{\ \ m}$$
$$+g^{im}g^{jn}a^{l}_{mn,l}+g^{im}g^{ln}a^{j}_{lm,n})\xi_i\xi_j$$
$$+(g^{ik}a^{j}+a^{j\cdot ki}+2a^{i\cdot jk}+3a^{k\cdot ij})\xi_i\xi_{j,k}+g^{im}g^{jn}\xi_{n,i}\xi_{m,j}$$
$$+g^{im}g^{jn}\xi_{m,i,j}\xi_n]dV=0.$$

Next, applying it to the vector $(g^{im}\xi_m)_{,i}\, g^{jn}\xi_n$, we obtain

$$\int[(2a^{i\cdot m}_{\ \ n}a^{n\cdot j}_{\ \ m}+a^{i}a^{j}+g^{mn}g^{li}a^{j}_{mn,l})\xi_i\xi_j$$
$$+(a^{j\cdot ki}+a^{k\cdot ij}+2g^{jk}a^{i}+g^{ki}a^{j})\xi_i\xi_{j,k}$$
$$+g^{im}g^{jn}\xi_{m,i,j}\xi_n+g^{im}g^{jn}\xi_{m,i}\xi_{n,j}]dV=0,$$

and consequently, from these, we have

(17) $$\int[((\Gamma^{ij}+2a^{m\cdot i}_{\ \ n}a^{n\cdot j}_{\ \ m}+a^{i\cdot m}_{\ \ n}a^{n\cdot j}_{\ \ m}+a^{m}a^{i\cdot j}_{\ \ m}+a^{i\cdot m}_{\ \ n}a^{j\cdot n}_{\ \ m}-a^{i}a^{j}$$
$$+g^{im}g^{jn}a^{l}_{mn,l}+g^{ln}g^{mi}(a^{j}_{lm,n}-a^{j}_{ln,m}))\xi_i\xi_j$$
$$+2(a^{ij\cdot k}+a^{k\cdot ij}-g^{jk}a^{i})\ \xi_i\xi_{j,k}$$
$$+g^{im}g^{jn}\xi_{i,n}\xi_{m,j}-g^{im}g^{jn}\xi_{m,i}\xi_{n,j}]dV=0.$$

If the vector ξ_i is general harmonic and the matrix (12) defines a non-negative quadratic form, by virtue of Theorem 3, we have

(18) $$2\int a^{(m\cdot n)i}\xi_{m,n}\xi_i\, dV$$
$$=\int(g^{lm}g^{n(j}a^{k)}_{lm,n}+a^{m}a^{(j\cdot k)}_{\ \ \ m}+a^{j}a^{k}-g^{m(j}g^{k)n}a^{l}_{mn,l}$$
$$+2a^{m\cdot(j}_{\ \ n}a^{|n|\cdot k)}_{\ \ \ \ m})\,\xi_j\xi_k\, dV=0.$$

If the vector ξ_i is general Killing and the matrix (14) defines a non-positive quadratic form, by virtue of Theorem 5, we have

(19) $$\int(g^{m(j}g^{k)n}a^{l}_{mn,l}-g^{lm}g^{n(j}a^{k)}_{lm,n}+2a^{m\cdot(j}_{\ \ n}a^{|n|\cdot k)}_{\ \ \ \ m}+2a^{m\cdot(j}_{\ \ n}a^{k)\cdot n}_{\ \ \ m}$$
$$+a^{m}a^{(j\cdot k)}_{\ \ \ m}+2a^{(j\cdot|m|}_{\ \ \ \ n}a^{k)\cdot n}_{\ \ \ m}-a^{(j}a^{k)})\xi_j\xi_k\, dV=0$$

In Riemannian mainfold and metric manifold with torsion, the theorems corresponding to theorems 3 and 5 can be proved by virtue of the integral (16), but, in our manifold, they can be proved when, and only when, (18) and (19) are held respectively.

5. The conditions that a vector be general harmonic and Killing.

Under the same assumptions in 4, for any vector ξ_i, it satisfies

$$\int [g^{jk}(\xi^m \xi_m)_{,j}]_{,k}\, dV=0,$$

and calculating this, we have

$$\begin{aligned}\int [(a^m a^{j\cdot k}_{\cdot m}+a^{j\cdot m}_{\cdot n} a^{k\cdot n}_{\cdot m}+a^{m\cdot j}_{\cdot n} a^{k\cdot n}_{\cdot m}+g^{mn} g^{jl} a^k_{lm,n})\xi_j \xi_k \\ +(2a^{m\cdot ni}+g^{im} a^n+2a^{i\cdot mn})\xi_i \xi_{m,n} \\ +g^{jk} g^{mn} \xi_{m,j,k} \xi_n+g^{jk} g^{mn} \xi_{m,j} \xi_{n,k}]dV=0.\end{aligned}$$

Subtracting this from (17), we have

$$\begin{aligned}\int [g^{mn} g^{jk} \xi_{m,j,k} \xi_n-(\Gamma^{ij}+2a^{m\cdot i}_{\cdot n} a^{n\cdot j}_{\cdot m}-a^i a^j+g^{im} g^{jn} a^l_{mn,l} \\ -g^{lm} g^{in} a^j_{lm,n})\xi_i \xi_j+(2a^{m\cdot ni}-2a^{n\cdot mi}+g^{im}a^n+2g^{mn}a^i)\xi_i\xi_{m,n} \\ +\frac{1}{2}g^{jk} g^{mn}(\xi_{m,j}-\xi_{j,m})(\xi_{n,k}-\xi_{k,n})+g^{jm} \xi_{m,j}\, g^{kn}\xi_{n,k}]dV=0.\end{aligned}$$

This equality holds always under the given assumptions. Hence, if the vector ξ_i satisfies

$$\begin{aligned}(20)\qquad g^{jk} \xi_{m,j,k}=\Gamma^i{}_m \xi_i+(2a^l_{mn} a^{n\cdot j}_{\cdot l}-a_m a^j+g^{jn} a^l_{mn,l}-g^{kl} a^j_{kl,m})\xi_j \\ -(2a^{j\cdot k}_{\cdot m}-2a^{k\cdot j}_{\cdot m}+\delta^j_m a^k+2g^{jk} a_m)\xi_{j,k},\end{aligned}$$

then we have

$$(21)\qquad \xi_{m,j}=\xi_{j,m}, \qquad g^{jm} \xi_{m,j}=0,$$

and consequently, the vector ξ_i is general harmonic. Furthermore, if ξ_i is general harmonic, then (7) is satisfied, and in order that (20) is held, it is necessary to satisfy

$$\begin{aligned}(22)\qquad (\delta^j_m a^k-2a^{j\cdot k}_{\cdot m})\, \xi_{j,k}=(2a^l_{mn} a^{n\cdot j}_{\cdot l}-a_m a^j+2g^{jn} a^l_{mn,l} \\ -g^{kl} a^j_{kl,m})\xi_j,\end{aligned}$$

and thus we have

THEOREM 6. *In a compact and oriented manifold with symmetric affine connection, if the vector* ξ_i *satisfies* (20), *then it is general harmonic, and conversely, when the general harmonic vector satisfies* (22), *it holds* (20).

In Riemannian manifold and the metric manifold with torsion, the equation corresponding to (20) is necessary and sufficient in order that a vector be harmonic.

Similarly we find

$$\begin{aligned}\int[&g^{mn} g^{jk} \xi_{n,j,k} \xi_n + (\Gamma^{ij} + 2a^{i\cdot m}_{\ \ n} a^{j\cdot m}_{\ \ n} + 2a^{i\cdot m}_{\ \ n} a^{n\cdot j}_{\ \ m} + 2a^{i\cdot j}_{\ \ n} a^{n\cdot j}_{\ \ m}\\ &+ 2a^m a^{i\cdot j}_{\ \ m} + g^{im} g^{nj} a^l_{mn,l} + g^{il} g^{nn}(2a^j_{ln,n} - a^j_{mn,l}))\xi_i \xi_j\\ &+ (4a^{i\cdot jk} + 2a^{j\cdot ki} + 2a^{k\cdot ij} + g^{ij} a^k - 2g^{jk} a^i)\xi_i \xi_{j,k}\\ &+ \frac{1}{2} g^{im} g^{jn}(\xi_{i,n} + \xi_{n,i})(\xi_{j,m} + \xi_{m,j}) - g^{im}\xi_{m,i} g^{jn} \xi_{n,j}]dV = 0,\end{aligned}$$

and if the vector ξ_i satisfies

$$\begin{aligned}(23)\qquad g^{jk} \xi_{m,j,k} = -\Gamma^i_m \xi_i - (&2a_{n\cdot\cdot n}^{\ \ \ l} a^{j\cdot n}_{\ \ l} + 2a_{m\cdot\cdot n}^{\ \ \ l} a^{j\cdot n}_{\ \ l} + 2a^l_{mn} a^{n\cdot j}_{\ \ l}\\ &+ 2a^n a_{m\cdot\cdot n}^{\ \ \ j} + g^{jn} a^l_{mn,l} + g^{ln}(2a^j_{ln,n} - a^j_{ln,m}))\xi_j\\ &- (4a_{m\cdot}^{\ \ jk} + 2a^{j\cdot k}_{\ \ m} + a^{k\cdot j}_{\ \ m} - \delta^j_m a^k)\xi_{j,k} = 0\end{aligned}$$

and

$$(24)\qquad g^{im} \xi_{m,i} = 0$$

then we have

$$\xi_{j,m} + \xi_{m,j} = 0,$$

and thus, the vector ξ_i is general Killing. Furthermore, if ξ_i is general Killing, then (24) is evidently satisfied, and in order that (23) is held, it is necessary to satisfy

$$\begin{aligned}(25)\qquad \delta^j_m a^k \xi_{j,k} + [2(&a_{m\cdot\cdot n}^{\ \ \ l} a^{j\cdot n}_{\ \ l} + a^l_{nn} a^{j\cdot n}_{\ \ l} + a^l_{mn} a^{n\cdot j}_{\ \ l} + a^n a_{m\cdot n}^{\ \ \ j})\\ &+ g^{jn} a^l_{mn,l} + g^{ln}(2a^j_{ln,n} - a^j_{ln,m})]\xi_j = 0.\end{aligned}$$

Hence we have

THEOREM 7. *In a compact and oriented manifold with symmetric affine connection, if the vector* ξ_i *satisfies* (23) *and* (24), *then it is* *general Killing, and conversely, wehn the Killing vector satisfies* (25) *it holds* (23) *and* (24).

Jan. 1959
Mathematical Department,
Kyungpook University.

REFERENCES

[1] K. Yano and S. Bochner; *Curvature and Betti Numbers* . Princeton 1953.

[2] L. P. Eisenhart; *Non-Riemannian Geometry*. Coll. of Amer. Math. Soci. 1926.