THE TOPOLOGIES OF PARTIALLY ORDERED SET WITH FINITE WIDTH

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1. Introduction

Recently, E. S. Wolk [1] has introduced the concept of *order-compatible* topology for partially ordered sets (poset), and has proved the following theorm:

If P is a poset of finite width, then P possesses a unique order-compatible topology. Furthermore, with respect to this topology, P is a Hausdorff space.

In this paper, we shall introduce the new simple topology for poset which we call *C-topology* and compare with certain topolgies and with order-compatible topology ($=OCT$) on the same poset P.

Let us introduce, first, the following definition.

DEFINITON. Let M be a subset of P, we shall say that a is *C-limiting point* of M if and only if there is such a chain C containing no a that C is included in M and $\wedge C=a$, or $\vee C=a$ in P

And let $\overline{M}=M\cup M'$ (M' is the set of all C-limiting point of M), we call $\overline{M}$ the *closure* of M. Also define a subset M of poset P to be *closed* in the C-topology if and only if $\overline{M}=M$, let CT denote the C-topology.

To this end, let us call a subset S of P *up-direct* (*down-derect*) if and only if for all $x\in S$, and $y\in S$, there exists $z\in S$ with $z\geqq x$, $z\geqq y$ ($z\leqq x$, $z\leqq y$) [1].

Also we shall call a subset K of P *Dedekind-closed* if and only if whenever S is an up direct subset of K and $y=\vee S$, or S is a down direct of K and $y=\wedge S$, we have $y\in K$.

We define that two elements x and y are *incomparable* if and only if neither $x\leqq y$ nor $x>y$, and we denote $x\# y$.

Let us say a subset D of P *diverse* if and only if $x \in D$, $y \in D$ and $x \neq y$ imply $x \# y$. And define the *width* of P to be the l. u. b. of the set $\{k \mid k$ is the cardinal number of a diverse subset of $P\}$.

In § 3, We shall then prove that in a poset of finite width the interval topology and C-topology are equivalent. And we shall see, by lemma 1, the interval topology, any compatible topology, Dedekind topology, order topology and C-topology are all equivalent.

Finally, we shall give some corollaries.

2. Definitions and Notations.

Let $\{x_\alpha\}$ be any direct set in a subset X of a poset P. We define x_α order-converges to a to mean $a = \bigvee_\alpha \{\bigwedge_{\beta \geqq \alpha} x_\beta\} = \bigwedge_\alpha \{\bigvee_{\beta \geqq \alpha} x_\beta\}$ in the completion of P by cuts. As usual, we define a subset X of a poset to be *closed* in the order-topology, denotes OT, if and only if $\{x_\alpha\} \subseteq X$ and x_α order-converges to a implies $a \in X$.

In P, we write

$$M(x) = \{a \mid x \leqq a\},\ L(x) = \{a \mid a \leqq x\} \text{ for each } x \in P.$$

The interval topology [2] is that topology generated by taking all of the set $\{M(x), L(x) \mid x \in P\}$ as a subbasis for the closed sets. Let us denote IT. It is well-known that the topology IT is weaker than the topology OT, and every poset P is Hausdorff space in topology OT.

If R_1 and R_2 are any topologies on P, we define $R_1 \leqq R_2$ to mean that every R_1-closed set is R_2-closed. It is then easy that $OT \leqq CT$, in fact, let subset of X of P be a closed set in OT. If the point a is a C-limiting point of X, then there exists chain $C = \{c_\alpha\}$ in X such that $\bigwedge c_\alpha = a$, or $\bigvee c_\alpha = a$. Assume $\bigwedge c_\alpha = a$, where $c_\alpha \geqq c_\beta$ in P if and only if $\alpha \leqq \beta$ in a chain directed indices class Γ, then $a = \bigwedge_\alpha \{\bigvee_{\beta \geqq \alpha} c_\beta\} = \bigvee_\alpha \{\bigwedge_{\beta \geqq \alpha} c_\beta\}$ in completion of P by cuts.

We define a topology DT on P whose closed sets are precisely the Dedekind closed sebsets of P. Then we see that $DT \leqq CT$. In fact, let subset X of P be a closed set in DT, Then X is Dedekint closed. If the point a is a C-limiting point of X such that $\bigwedge C = a$ (or $\bigvee C = a$) for

a chain $C\subseteq X$, since chain C is an upper direct set, then $a\in X$. Similarly, for $\bigvee C=a$. By the lemma in [1], $IT\leqq OCT\leqq DT$.
Hence, we have the following lemma

LEMMA 1. $IT\leqq OCT\leqq DT\leqq CT$ *and* $IT\leqq OT\leqq CT$

3. The principal theorems

The author has proved the following theorem [3]

A necessary and sufficient condition for an element a of a poset P to be isolated to subset X in IT is that, for the element a there exist the finite subsets A and B of P such that

(*i*) $A=\{x \mid x\# a,$ *or* $x>a\}$, $B=\{y \mid y\# a,$ *or* $y<a\}$,

(*ii*) $(M(x))_{x\in A}$, $(L(y))_{y\in B}$ *are a covering of* $X-a$ (*i.e.*, *Union of the collection includes* $X-a$)

And we now prove the following

THEOREM 1. *Let P be a poset of finite width, then a necessary and sufficient condition for an element a to be isolated to subset X in CT is that, for the element a there exist the finite subsets A and B of P satisfying* (*i*) *and* (*ii*).

PROOF In view of the above theorem and lemma, we need only to prove the necessary.

Let $S=\{x \mid x\in X-a$ and $x>a\}$, Let k be width of P. The number of all the least elements $l_i\in S$ of a maximal chain of S are k at most, let $E=\{l_1, l_2, \cdots l_m\}$.

Let C be arbitrary maximal chain containing no the least elemnet in S if exist. There exists at least one point $x\in P-S$ with $a<x<c$ for all $c\in C$. In fact, let us suppose that our requierment is false, then $\bigwedge C=a$, i.e., a is a C-limiting point of X which is contrary.
Let C_i be a maximal chain of S containing no the least element in S, and let x_i be the point of $P-S$ such that $a<x_i<c$ for all $c\in C_i$ ($i=1,2,\cdots$). If $x_1>x_2>x_3>\cdots$ and when there exists the point $c_i\in C_i$

with $c_z \bar{\in} M(x_{z-1})$, then such x_z s are at most k. In fect, since $c_2 \bar{\in} M(x_1)$, there exists at least one $y_1 \in C_1$ with $c_2 \# z_1$ for all $z_1 \leq y_1$ in C_1. Otherwise, $c_2 \leq c$ for all $c \in C_1$, but $c_2 \bar{\in} C_1$, hence we have contradiction to the maximality of C_1. Similarly, for $c_3 \bar{\in} M(x_2)$, we have $y_2 \in C_2$ with $c_3 \# z_2$ for all $z_2 \leq y_2$, in C_2. And since $c_3 \bar{\in} M(x_1)$, we have also $y_{13} \in C_1$ with $c_3 \# z_{13}$ for all $z_{13} \leq y_{13}$ in C_1. If now $c_2 \leq y_2$ in C_2 and $y_{13} \leq y_1$ in C_1, then $\{y_{13}, c_2, c_3\}$ is diverse set. If $y_1 < y_{13}$ in C_1, then $\{y_1, c_2, c_3\}$ is diverse set.

If $c_2 > y_2$ in C_2, we can see easily that $y_2 \bar{\in} M(x_1)$ and we have $y_{12} \in C_1$ with $y_2 \# z_{12}$ for all $z_{12} \leq y_{12}$ in C_1. And if $y_{12} \leq y_{13}$ in C_1, we have $\{y_{12}, y_2, c_3\}$ is diverse set. If $y_{13} < y_{12}$ in C_1, $\{y_{13}, y_2\ c_3\}$ is diverse set. Hence for any case, we have a diverse set of three elements, if exist the three maximal chains with the above porperties. Therefore, relative comparable x_z s are at most k, since if there exist more than k elements x_z, then continuing the above construction leads to a diverse set with more than k elements.

Let $F = \{m_1, \cdots m_p\}$ be the set of all the minimal elements of relative comparable $x_z s$. Clearly, F is a finite set, $(p \leq k)$. Thus we have $\cup_{x \in E \cup F} M(x) \supseteq S$.

Let $S' = \{y \mid y \in X - a,$ and $y < a\}$. Dually, we have the finite subsets $G = \{g_1, \cdots, g_n\}$, $H = \{n_1, \cdots, n_q\}$ of P such that $\cup_{y \in G \cup H} L(y) \supseteq S'$ And let $N = \{a, u_1, \cdots, u_l\}$ be a maximal diverse set containing a. It is clear that if put $A = E \cup F \cup (N - a)$, $B = G \cup H \cup (N - a)$, then

$$(\cup_{x \in A} M(x)) \cup (\cup_{y \in B} L(y)) \supseteq X - a.$$

Hence the proof is complete.

If P be a poset of finite width, a point a is isolated to a subset M of P in CT if and only if a is also isolated to M in IT. That is, a point b is a limiting point of M in IT if and only if b is also a limiting point of M in CT.

Thus, by lemma, we have

THEOREM 2. *If P is a poset of finite width. then topologies IT, any OCT, DT, OT, and CT are all equivalent.*

In the special case when P is a lattice, we have conversely the following.

LEMMA 2 *Let P be a lattice. If topology IT and topology CT are equivalent, then P has no infinite diverse set.*

In fact, let us suppose that the lemma is false, and let D be an infinte diverse set, then D is clearly closed set in CT. Hence, by Hypothese, D is also a closed set in IT. Since P is a lattice, we may find the finite closed intervals $I_1, I_2, \cdots I_n$ such that $D = I_1 \cup I_2 \cup \cdots \cup I_n$ [4]. But there is no such finite closed interval in infinite diverse set D.

We now have the folloing theorem

THEOREM 3. *Let P be a lattice with least and greatest elements. P has no infinite diverse set if and only if topology IT and topology CT are equivalent.*

COROLLARY 1. *Let P be a poset of finite width, then P possesses a unique ordered-compatible topology. Furthermor , with respect to this topology, P is a Hausdorff space.* (This theorem has already been solved by E. S. Wolk [1]).

In fact, topology OT is Hausdorff, and OT and OCT are equivalent.

COROLLARY 2. *Let P be a poset of finite width and let* $\{x_\alpha\}$ *be a direct set such that* $a = \bigwedge_{\alpha} \{ \bigvee_{\beta \geqq \alpha} x_\beta \} = \bigvee_{\alpha} \{ \bigwedge_{\beta \geqq \alpha} x_\beta \}$, *then there exists a chain* $C \subseteq \{x_\alpha\}$ *such that* $\bigwedge C = a$, or $\bigvee C = a$.

COROLLARY 3, *Let P be a complete lattice with no infinite diverse set, then P is compact Hausdorff space in topologies IT, OCT, DT, OT, CT.*

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