ON CONHARMONIC TRANSFORMATIONS IN HERMITIAN AND KAEHLERIAN SPACES

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Introduction.

The present auther wishes to study a special conformal transformation which we shall call conharmonic in a Hermitian space H_n and a Kaehlerian space K_n. The conharmonic transformations in the Riemannian spaces were already studied by Ishii [1][1)]

The purpose of the present paper is to find a tensor whose invariability under a special conformal transformation is a necessary and sufficient condition that H_n and K_n are conharmonic. (Theorem 4)

1. Conharmonic transformations.

For the Hermitian space H_n, we denote the metric tensor, Christoffel symbols, curvature tensor by the notations G_{ij}, E^i_{jk}, $H^i{}_{jkl}$ [2)] respectively, and for the Kaehlerian space K_n, referred to complex analytic coordinates $z^i=(z^\alpha, \bar{z}^\alpha)$ $(\bar{z}^\alpha=z^{\bar{\alpha}})$, we denote by notations g_{ij}, Γ^i_{jk}, $R^i{}_{jkl}$ respectively, then $g_{\alpha\beta}=g_{\bar{\alpha}\bar{\beta}}=0$, $G_{\alpha\beta}=G_{\bar{\alpha}\bar{\beta}}=0$ and g_{ij}, G_{ij} are the symmetric and self-adjoint tensors.

We shall consider a special conformal transformation

$$G_{\alpha\bar{\beta}}=\phi^2 g_{\alpha\bar{\beta}} \tag{1. 1}$$

where $\phi(z, \bar{z})$ is a real valued function in connection with

$$g_{\alpha\bar{\beta}}=\frac{\partial^2\phi}{\partial z^\alpha \partial \bar{z}^\beta}=\partial_{\bar{\beta}}\,\partial_\alpha\phi$$

Let A be a harmonic function which we shall define by

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1) Numbers in brackets refer to the references at the end of the paper.

2) In the following the Latin indices $i, j, k, \cdots$ are supposed to run over the range $1, 2, \cdots, n, \bar{1}, \bar{2}, \cdots, \bar{n}$ and the Greek indicies $\alpha, \beta, \gamma, \cdots$ take the values $1, 2, \cdots, n$ and consequently $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \cdots$ run over the range of the symbols $\bar{1}, \bar{2}, \cdots, \bar{n}$.

(1. 2) $g^{\alpha\bar{\beta}} A_{,\alpha\bar{\beta}}=0$

and let seek the condition upon ϕ in order that the function defined by

(1. 3) $B=\phi^{2m} A$

may become a harmonic function with respect to the tensor $G_{\alpha\bar{\beta}}$, i. e.

$$G^{\alpha\bar{\beta}} B_{|\alpha\bar{\beta}}=0$$

where "," and "|" denote covariant differentiation with respect to the tensor $g_{\alpha\bar{\beta}}$ and $G_{\alpha\bar{\beta}}$ respectively, and m is a suitable constant.

By the relations

$$E^{\alpha}_{\beta\gamma}=\frac{1}{2}G^{\alpha\bar{\rho}}(\partial_{\gamma} G_{\beta\bar{\rho}}+\partial_{\beta} G_{\gamma\bar{\rho}}) \qquad \text{(conj.)}$$

$$E_{\beta}{}^{\alpha}{}_{\bar{\gamma}}=\frac{1}{2}G^{\alpha\bar{\rho}}(\partial_{\bar{\gamma}}G_{\beta\bar{\rho}}-\partial_{\bar{\rho}} G_{\beta\bar{\gamma}}) \qquad \text{(conj.)}$$

and Kaehlerian condition, we find the following relations

(1. 4) $E^{\alpha}_{\beta\gamma}=\Gamma^{\alpha}_{\beta\gamma}+\delta^{\alpha}_{\beta} (\log\phi)_{,\gamma}+\delta^{\alpha}_{\gamma} (\log\phi)_{,\beta}$ (conj.)

(1. 5) $E^{\alpha}_{\beta\bar{\gamma}}=\delta^{\alpha}_{\beta} (\log\phi)_{,\bar{\gamma}} -g^{\alpha\bar{\rho}} g_{\beta\bar{\gamma}} (\log\phi)_{,\bar{\rho}}$ (conj.)

therefore

(1. 6)
$$\begin{aligned} G^{\alpha\bar{\beta}} B_{|\alpha\bar{\beta}} &=G^{\alpha\bar{\beta}}[\partial_{\bar{\beta}}\, \partial_{\alpha} B-(\partial_{\rho}B)\, E^{\rho}_{\alpha\bar{\beta}}-(\partial_{\bar{\sigma}}B)\, E^{\bar{\sigma}}_{\alpha\bar{\beta}}] \\ &=[2m(2m-3+2n)\phi^{2(m-2)}\, g^{\rho\bar{\sigma}}\, \phi_{\rho}\, \phi_{\bar{\sigma}}+2mn\, \phi^{2m-3}]A \\ &\quad +(2m+n-1)\phi^{2m-3}\, g^{\alpha\bar{\beta}}\, (\phi_{\alpha}\, A_{,\bar{\beta}}+\phi_{\bar{\beta}}\, A_{,\alpha}) \\ &\quad +\phi^{2(m-1)}\, g^{\alpha\bar{\beta}}\, A_{,\alpha\bar{\beta}}=0 \end{aligned}$$

where

$$\phi_{\alpha}=\phi_{,\alpha}=\partial_{\alpha}\, \phi \qquad \text{(conj.)}$$

If we determine m by

(1. 7) $m=\dfrac{1-n}{2}$

we have from (1. 2)

$$\phi^{-(n+3)}\,[(n-2)\, g^{\rho\bar{\sigma}}\phi_{\rho}\, \phi_{\bar{\sigma}}+n\phi]\, A=0$$

Hence the required condition upon ϕ is

(1. 8) $(n-2)\, g^{\rho\bar{\sigma}}\, \phi_{\rho}\, \phi_{\bar{\sigma}}+n\phi=0$

We shall call the conformal transformation(1. 1) satisfying (1. 8) conharmonic transformation [1].

Now, let us consider a vector A_{α} of K_n and suppose that by a conformal transformation (1. 1) A_{α} be transformed into B_{α} defined by

(1. 9) $$B_\alpha = A_\alpha - (\log\phi)_{,\alpha}$$

then

$$G^{\alpha\bar\beta} B_{\alpha|\bar\beta} = \phi^{-2}[g^{\alpha\bar\beta} A_{\alpha,\bar\beta} - n\phi^{-1} + (n-1)\phi^{-1} g^{\alpha\bar\beta}(\phi_\alpha A_{\bar\beta} + A_\alpha \phi_{\bar\beta}) + (3-2n)\phi^{-2} g^{\alpha\bar\beta}\phi_\alpha\phi_{\bar\beta}]$$

$$G^{\alpha\bar\beta} B_\alpha B_{\bar\beta} = (n-1)\phi^{-2} g^{\alpha\bar\beta}[A_\alpha A_{\bar\beta} - \phi^{-1}(\phi_\alpha A_{\bar\beta} + A_\alpha \phi_{\bar\beta}) + \phi^{-2} g^{\alpha\bar\beta}\phi_\alpha\phi_{\bar\beta}]$$

therefore

$$G^{\alpha\bar\beta}[B_{\alpha|\bar\beta} + (n-1) B_\alpha B_{\bar\beta}] = \phi^{-2} g^{\alpha\bar\beta}[A_{\alpha,\bar\beta} + (n-1)A_\alpha A_{\bar\beta}] - \phi^{-4}[(n-2) g^{\alpha\bar\beta}\phi_\alpha\phi_{\bar\beta} + n\phi]$$

thus we have the following theorem which is similar to Lemma in pp. 77 of [1].

THEOREM 1. *A necessary and sufficient condition that the conformal transformation* (1. 1) *be conharmonic is that a vector field* A_α *of* K_n *is transformed into a vector field* B_α *of* H_n *defined by* (1. 9) *and satisfies the condition*

$$G^{\alpha\bar\beta}[B_{\alpha|\bar\beta} + (n-1)B_\alpha B_{\bar\beta}] = \phi^{-2} g^{\alpha\bar\beta}[A_{\alpha,\bar\beta} + (n-1)A_\alpha A_{\bar\beta}]$$

2. Conharmonic invariant tensors.

By the transformation (1. 1) the non-zero components of curvature tensor of H_n are represented by the following forms

(2. 1) $$H^\alpha{}_{\beta\gamma\delta} = \phi^{-1}\delta^\alpha_\gamma(\phi_{\beta,\delta} - 2\phi^{-1}\phi_\beta\phi_\delta) - \phi^{-1}\delta^\alpha_\delta(\phi_{\beta,\gamma} - 2\phi^{-1}\phi_\beta\phi_\gamma) \quad \text{(conj.)}$$

(2. 2) $$H^\alpha{}_{\beta\bar\gamma\delta} = R^\alpha{}_{\beta\bar\gamma\delta} - 2\phi^{-1}\delta^\alpha_\delta(g_{\beta\bar\gamma} - \phi^{-1}\phi_\beta\phi_{\bar\gamma}) + 2\phi^{-2} g_{\beta\bar\gamma}(g^{\alpha\bar\sigma}\phi_\delta\phi_{\bar\sigma} - \delta^\alpha_\delta g^{\rho\bar\sigma}\phi_\rho\phi_{\bar\sigma}) \quad \text{(conj.)}$$

(2. 3) $$H^\alpha{}_{\bar\beta\gamma\delta} = 2\phi^{-1}\delta^\alpha_\gamma[g_{\delta\bar\beta} - \phi^{-1}\phi_\delta\phi_{\bar\beta} + \phi^{-1} g_{\delta\bar\beta} g^{\rho\bar\sigma}\phi_\rho\phi_{\bar\sigma}] - 2\phi^{-1}\delta^\alpha_\delta[g_{\gamma\bar\beta} - \phi^{-1}\phi_\gamma\phi_{\bar\beta} + \phi^{-1} g_{\gamma\bar\beta} g^{\rho\bar\sigma}\phi_\rho\phi_{\bar\sigma}] + 2\phi^{-2} g^{\alpha\bar\sigma}[g_{\gamma\bar\beta}\phi_\delta - g_{\delta\bar\beta}\phi_\gamma]\phi_{\bar\sigma} \quad \text{(conj.)}$$

(2. 4) $$H^\alpha{}_{\bar\beta\gamma\bar\delta} = \phi^{-1}\delta^\alpha_\gamma(\phi_{\bar\beta,\bar\delta} - 2\phi^{-1}\phi_{\bar\beta}\phi_{\bar\delta}) - \phi^{-1} g_{\gamma\bar\beta}[(\partial_{\bar\delta} g^{\alpha\bar\sigma})\phi_{\bar\sigma} + g^{\alpha\bar\sigma}((\partial_{\bar\delta}\phi_{\bar\sigma}) - 2\phi^{-1}\phi_{\bar\delta}\phi_{\bar\sigma})] \quad \text{(conj.)}$$

(2. 5)
$$H^{\alpha}{}_{\beta\bar{\gamma}\bar{\delta}}=\phi^{-1} g_{\beta\bar{\delta}}[\partial_{\bar{\gamma}}(g^{\alpha\bar{\sigma}})\phi_{\bar{\sigma}}+g^{\alpha\bar{\sigma}}(\partial_{\bar{\gamma}}\phi_{\bar{\sigma}}) -2\phi^{-1}g^{\alpha\bar{\sigma}}\phi_{\bar{\gamma}}\phi_{\bar{\sigma}}]+\phi^{-1} g_{\beta\bar{\gamma}}[\partial_{\bar{\delta}}(g^{\alpha\bar{\sigma}})\phi_{\bar{\sigma}} +g^{\alpha\bar{\sigma}}(\partial_{\bar{\delta}}\phi_{\bar{\sigma}})-2\phi^{-1} g^{\alpha\bar{\sigma}}\phi_{\bar{\delta}}\phi_{\bar{\sigma}}] \qquad \text{(conj.)}$$

Since

$$H^{\alpha}{}_{\beta\bar{\gamma}\alpha}=R_{\beta\bar{\gamma}}-2n(\phi^{-1}g_{\beta\bar{\gamma}}-\phi^{-2}\phi_{\beta}\phi_{\bar{\gamma}}) +2(1-n)\phi^{-2} g_{\beta\bar{\gamma}} g^{\rho\bar{\sigma}}\phi_{\rho}\phi_{\bar{\sigma}}$$

$$H^{\bar{\alpha}}{}_{\beta\bar{\gamma}\bar{\alpha}}=2\phi^{-1}(1-n) g_{\beta\bar{\gamma}}+2(n-2)\phi^{-2}\phi_{\beta}\phi_{\bar{\gamma}} -2(n-2)\phi^{-2}g_{\beta\bar{\gamma}} g^{\rho\bar{\sigma}}\phi_{\rho}\phi_{\bar{\sigma}}$$

we have

(2. 6)
$$H_{\beta\bar{\gamma}}=H^{\alpha}{}_{\beta\bar{\gamma}\alpha}+H^{\bar{\alpha}}{}_{\beta\bar{\gamma}\bar{\alpha}} =R_{\beta\bar{\gamma}}+2(1-2n)\phi^{-1} g_{\beta\bar{\gamma}}+4(n-1)\phi^{-2}\phi_{\beta}\phi_{\bar{\gamma}} +2(3-2n)\phi^{-2} g_{\beta\bar{\gamma}} g^{\rho\bar{\sigma}}\phi_{\rho}\phi_{\bar{\sigma}}$$

and if transformation (1. 1) be conharmonic, then by (1. 8)

(2. 7)
$$H_{\beta\bar{\gamma}}=R_{\beta\bar{\gamma}}+4(1-n)\phi^{-2}\left[\frac{\phi}{2-n}g_{\beta\bar{\gamma}}-\phi_{\beta}\phi_{\bar{\gamma}}\right]$$

and on multiplying this by $G^{\beta\bar{\gamma}}=\phi^{-2} g^{\beta\bar{\gamma}}$ and contracting, we get

(2. 8)
$$H=\phi^{-2} R \qquad \text{where} \qquad R=2g^{\alpha\bar{\beta}}R_{\alpha\bar{\beta}}, \qquad H=2G^{\alpha\bar{\beta}}H_{\alpha\bar{\beta}}$$

Conversely if (2. 8) hold, then from (2. 6) we can see that (1. 8) is satisfied, thus we have the following [1]

THEOREM 2. *A necessary and sufficient condition that a conformal transformation* (1. 1) *be conharmonic is that it satisfies the condition* (2. 8).

If the Ricci tensor is invariant under the conharmonic transformation (1. 1), i.e. $H_{\beta\bar{\gamma}}=R_{\beta\bar{\gamma}}$ then from (2. 7) we have

$$\phi\, g_{\beta\bar{\gamma}}+(n-2)\phi_{\beta}\phi_{\bar{\gamma}}=0$$

but by the relation

$$\phi(\partial_{\rho} g_{\beta\bar{\gamma}})=(1-n)[(\partial_{\rho}\phi_{\beta})\phi_{\bar{\gamma}}+\phi_{\beta} g_{\rho\bar{\gamma}}]-g_{\beta\bar{\gamma}}\phi_{\rho}$$

and Kaehlerian condition, we have $\phi_{\alpha}=0$ (conj.): thus we have the following

THEOREM 3. *There exists no such conharmonic transformation (1. 1) that the Ricci tensor is invariant.*

If transformation (1. 1) be conharmonic, then from (1. 8)

$$H^{\alpha}{}_{\beta\bar{\gamma}\delta}=R^{\alpha}{}_{\beta\bar{\gamma}\delta}+\frac{4}{n-2}\phi^{-1}\delta^{\alpha}_{\delta}g_{\beta\bar{\gamma}} +2\phi^{-2}(\delta^{\alpha}_{\delta}\phi_{\beta}\phi_{\bar{\gamma}}+g_{\beta\bar{\gamma}}g^{\alpha\bar{\sigma}}\phi_{\delta}\phi_{\bar{\sigma}}) \tag{2. 9}$$

and from (2. 7)

$$\frac{1}{2(n-1)}\delta^{\alpha}_{\delta}H_{\beta\bar{\gamma}}=\frac{1}{2(n-1)}\delta^{\alpha}_{\delta}R_{\beta\bar{\gamma}} -2\phi^{-2}\delta^{\alpha}_{\delta}[-\frac{\phi}{2-n}g_{\beta\bar{\gamma}}-\phi_{\beta}\phi_{\bar{\gamma}}]$$

$$\frac{1}{2(n-1)}G_{\beta\bar{\gamma}}G^{\alpha\bar{\sigma}}H_{\delta\bar{\sigma}}=\frac{1}{2(n-1)}g_{\beta\bar{\gamma}}g^{\alpha\bar{\sigma}}R_{\delta\bar{\sigma}} -2\phi^{-2}g_{\beta\bar{\gamma}}g^{\alpha\bar{\sigma}}[-\frac{\phi}{2-n}g_{\delta\bar{\sigma}}-\phi_{\delta}\phi_{\bar{\sigma}}]$$

Hence we have

$$H^{\alpha}{}_{\beta\bar{\gamma}\delta}-\frac{1}{2(n-1)}[\delta^{\alpha}_{\delta}H_{\beta\bar{\gamma}}+G_{\beta\bar{\gamma}}G^{\alpha\bar{\sigma}}H_{\delta\bar{\sigma}}] =R^{\alpha}{}_{\beta\bar{\gamma}\delta}-\frac{1}{2(n-1)}[\delta^{\alpha}_{\delta}R_{\beta\bar{\gamma}}+g_{\beta\bar{\gamma}}g^{\alpha\bar{\sigma}}R_{\delta\bar{\sigma}}] \tag{2.10}$$

therefore

$$C^{\alpha}{}_{\beta\bar{\gamma}\delta}=R^{\alpha}{}_{\beta\bar{\gamma}\delta}-\frac{1}{2(n-1)}[\delta^{\alpha}_{\delta}R_{\beta\bar{\gamma}}+g_{\beta\bar{\gamma}}g^{\alpha\bar{\sigma}}R_{\delta\bar{\sigma}}] \tag{2.11}$$

is invariant under the conharmonic transformation (1. 1).

Conversely if $C^{\alpha}{}_{\beta\bar{\gamma}\delta}$ be invariant under the transformation (1. 1), we have from (2. 6)

$$H^{\alpha}{}_{\beta\bar{\gamma}\delta}-R^{\alpha}{}_{\beta\bar{\gamma}\delta}=2\phi^{-2}[\delta^{\alpha}_{\delta}\phi_{\beta}\phi_{\bar{\gamma}}+2g_{\beta\bar{\gamma}}g^{\alpha\bar{\sigma}}\phi_{\delta}\phi_{\bar{\sigma}}] +\frac{2}{n-1}\phi^{-1}\delta^{\alpha}_{\delta}g_{\beta\bar{\gamma}}[(1-2n)+(3-2n)\phi^{-1}g^{\rho\bar{\sigma}}\phi_{\rho}\phi_{\bar{\sigma}}]$$

comparing this relation with the relation (2. 2) we have

$$\frac{2}{n-1}\phi^{-1}\delta^{\alpha}_{\delta}g_{\beta\bar{\gamma}}[n+(n-2)\phi^{-1}g^{\rho\bar{\sigma}}\phi_{\rho}\phi_{\bar{\sigma}}]=0$$

this relation is equivalent to (1. 8), thus we have the following

THEOREM 4. *A necessary and sufficient condition that the conformal transformation* (1. 1) *be conharmonic is that the tensor* $C^{\alpha}{}_{\beta\bar{\gamma}\delta}$ *is invariant under the transformation* (1. 1).

If $C^{\alpha}{}_{\beta\bar{\gamma}\delta}=0$, from (2. 11) we get by contraction

$$R_{\beta\bar{\gamma}}-\frac{R}{2(n-2)}g_{\beta\bar{\gamma}}=0$$

and on multiplying this by $g^{\beta\bar{\gamma}}$ and contracting, we get $R=0$, and by Theorem 2 we have the following

THEOREM 5. *If the transformation* (1. 1) *be conharmonic and* $C^{\alpha}{}_{\beta\bar{\gamma}\delta}$ *be a zero tensor in* K_n, *then* $R=0$ *and also* $H=0$ *in* H_n.

3. Conharmonic sectional curvatures

If the sectional curvature L of H_n [2] be same for all possible 2-dimensional section, then the curvature tensor must have the following form

$$H_{ijkl}=L(G_{jk}G_{il}-G_{jl}G_{ik})$$

but in the present case this reduce to

$$\text{(3. 1)}\qquad \begin{aligned} &H_{\bar{\alpha}\bar{\beta}\gamma\delta}=L(G_{\gamma\bar{\beta}}\,G_{\delta\bar{\alpha}}-G_{\gamma\bar{\alpha}}\,G_{\delta\bar{\beta}}) &&\text{(conj.)}\\ &H_{\alpha\bar{\beta}\gamma\bar{\delta}}=LG_{\alpha\bar{\delta}}\,G_{\gamma\bar{\beta}} &&\text{(conj.)}\end{aligned}$$

and on substituting this into $H_{\beta\bar{\gamma}}$, we find

$$\text{(3. 2)}\qquad H_{\beta\bar{\gamma}}=H^{\rho}{}_{\beta\bar{\gamma}\rho}+H^{\bar{\rho}}{}_{\beta\bar{\gamma}\bar{\rho}}=2(n-1)\,LG_{\beta\bar{\gamma}}$$

If K_n be a space of constant holomorphic curvature, we have (pp. 162 of [2])

$$\text{(3. 3)}\qquad R_{\alpha\bar{\beta}\gamma\bar{\delta}}-\frac{1}{2(n+1)}(g_{\alpha\bar{\beta}}R_{\gamma\bar{\delta}}+g_{\alpha\bar{\delta}}R_{\gamma\bar{\beta}}+g_{\gamma\bar{\delta}}R_{\alpha\bar{\beta}}+g_{\gamma\bar{\beta}}R_{\alpha\bar{\delta}})=0$$

$$\text{(3. 4)}\qquad R_{\beta\bar{\gamma}}=\frac{R}{2n}g_{\beta\bar{\gamma}}=\frac{R}{2n}\phi^{-2}\,G_{\beta\bar{\gamma}}$$

and by (2. 11) we have

$$\text{(3. 5)}\qquad g_{\alpha\bar{\beta}}\,R_{\gamma\bar{\delta}}+g_{\gamma\bar{\delta}}\,R_{\alpha\bar{\beta}}=2(n-1)(R_{\alpha\bar{\beta}\gamma\bar{\delta}}-C_{\alpha\bar{\beta}\gamma\bar{\delta}})$$

and on substituting this into (3. 3) and using the relations

$$R_{\alpha\bar{\beta}\gamma\bar{\delta}}=R_{\alpha\bar{\delta}\gamma\bar{\beta}}, \qquad C_{\alpha\bar{\beta}\gamma\bar{\delta}}=C_{\alpha\bar{\delta}\gamma\bar{\beta}}$$

we have

(3. 6) $$(n-3)R_{\alpha\bar{\beta}\gamma\bar{\delta}}=2(n-1)C_{\alpha\bar{\beta}\gamma\bar{\delta}}$$

Next, $C^{\bar{\alpha}}{}_{\bar{\beta}\gamma\bar{\delta}}$ is invariant under the conharmonic transformation (1. 1) therefore from (2. 10)

(3. 7) $$C_{\alpha\bar{\beta}\gamma\bar{\delta}}=\phi^{-2}[H_{\alpha\bar{\beta}\gamma\bar{\delta}}-\frac{1}{2(n-1)}(G_{\alpha\bar{\delta}}H_{\gamma\bar{\beta}}+G_{\gamma\bar{\beta}}H_{\alpha\bar{\delta}})]$$

and on substituting (3. 4) into (2. 10)

(3. 8) $$R_{\alpha\bar{\beta}\gamma\bar{\delta}}=\phi^{-2}[H_{\alpha\bar{\beta}\gamma\bar{\delta}}-\frac{1}{2(n-1)}(G_{\alpha\bar{\delta}}H_{\gamma\bar{\beta}}+G_{\gamma\bar{\beta}}H_{\alpha\bar{\delta}})+\frac{R\phi^{-2}}{2n(n-1)}G_{\alpha\bar{\delta}}G_{\gamma\bar{\beta}}]$$

By substituting (3. 7) and (3. 8) into (3. 6), and using(3. 1), (3. 2) and (2. 8) we find

(3. 9) $$L=\frac{1}{n(n+1)}H$$

therefore we obtain the following conclusion.

THEOREM 6. *If H_n whose sectional curvature L is same for all possible 2-dimensional sections is transformed, by the conharmonic transformation* (1. 1), *from K_n which is a space of constant holomorphic curvature, then we have the relation* (3. 9).

If we assume that at all points of the space H_n, the holomorphic sectional curvature is all the same, then we must have

(3.11) $$H_{\alpha\beta\bar{\gamma}\bar{\delta}}=L(G_{\beta\bar{\gamma}}G_{\alpha\bar{\delta}}-G_{\beta\bar{\delta}}G_{\alpha\bar{\gamma}}) \qquad \text{(conj.)}$$

(3.12) $$H_{\alpha\bar{\beta}\gamma\bar{\delta}}=-LG_{\alpha\bar{\delta}}G_{\gamma\bar{\beta}} \qquad \text{(conj.)}$$

(3.13) $$H_{\bar{\alpha}\beta\gamma\delta}=H_{\alpha\bar{\beta}\gamma\delta}=0 \qquad \text{(conj.)}$$

If (1. 1) be conharmonic transformation, then by (2. 1), (2. 2), (2. 3), (2. 5) and (1. 8) we obtain

$$(3.14)\qquad H_{\alpha\beta\bar{\gamma}\bar{\delta}}=\frac{4}{2-n}[\phi g_{\alpha\bar{\gamma}}g_{\beta\bar{\delta}}-g_{\alpha\bar{\delta}}g_{\beta\bar{\gamma}}]+2(g_{\alpha\bar{\delta}}\phi_\beta\phi_{\bar{\gamma}}-g_{\alpha\bar{\gamma}}\phi_\beta\phi_{\bar{\delta}}+g_{\beta\bar{\gamma}}\phi_\alpha\phi_{\bar{\delta}}-g_{\beta\bar{\delta}}\phi_\alpha\phi_{\bar{\gamma}})\qquad \text{(conj.)}$$

$$(3.15)\qquad H_{\alpha\bar{\beta}\gamma\bar{\delta}}=\phi^2R_{\alpha\bar{\beta}\gamma\bar{\delta}}+2(g_{\gamma\bar{\beta}}\phi_\alpha\phi_{\bar{\delta}}+g_{\alpha\bar{\delta}}\phi_\gamma\phi_{\bar{\beta}})-\frac{4}{2-n}\phi g_{\alpha\bar{\delta}}g_{\gamma\bar{\beta}}\qquad \text{(conj.)}$$

$$(3.16)\qquad H_{\bar{\alpha}\beta\gamma\delta}=\phi[g_{\gamma\bar{\alpha}}(\phi_{\beta,\delta}-2\phi^{-1}\phi_\beta\phi_\delta)-g_{\delta\bar{\alpha}}(\phi_{\beta,\gamma}-2\phi^{-1}\phi_\beta\phi_\gamma)]\qquad \text{(conj.)}$$

$$(3.17)\qquad H_{\alpha\bar{\beta}\gamma\delta}=\phi[g_{\delta\bar{\beta}}(\phi_{\alpha,\gamma}-2\phi^{-1}\phi_\alpha\phi_\gamma)-g_{\gamma\bar{\beta}}(\phi_{\alpha,\delta}-2\phi^{-1}\phi_\alpha\phi_\delta)]\qquad \text{(conj.)}$$

From (3.13) and (3.16) (or (3.17)) we get by contraction

$$\phi_{\alpha,\gamma}-2\phi^{-1}\phi_\alpha\phi_\gamma=0\qquad \text{(conj.)}$$

and, on substituting (3.12) into (3.15), and (3.11) into (3.14), and on multiplying by $G^{\alpha\bar{\delta}}G^{\gamma\bar{\beta}}$ and $G^{\alpha\bar{\delta}}G^{\beta\bar{\gamma}}$ respectively and contracting, we get by (1. 8)

$$L=-\phi^{-2}\frac{R}{2n^2}\qquad\text{and}\qquad n(n-1)L=0$$

hence we conclude the following

THEOREM 7. *There exists no Hermitian space H_n which is transformed by the conharmonic transformation* (1. 1) *from Kaehlerian space K_n, and whose holomorphic sectional curvature is all the same and not zero at all points.*

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