BOCHNER'S LEMMA ON THE CRAIG EXTENSOR FIELD

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1. Introduction.

In this paper I will apply the Bochner's lemma [1] on the extensor field which was introduced by H. V. Craig [2].

Let us consider an n-dimensional compact orientable Riemannian manifold V_n whose metric is defined by the definite quadratic form

$$ds^2 = g_{jk} dx^j dx^k$$

and whose element is an arc in V_n, given by n regular functions $x^i = x^i(t)$ where 't' is a fixed parameter. We shall use only different set of indicies to distinguish between different coordinate systems, eg. (x^i) and (x^r) etc, $(i, j, k, \cdots\cdots = 1, 2, \cdots, n)$. Moreover, we shall use bracketed Greek indicies α, β, γ etc. $(\alpha, \beta, \gamma, \cdots = 1, 2, \cdots\cdot, M)$ to indicate differentiations with respect to 't'.

By using the relation [2]

$$(1.\ 1) \qquad X^{(\alpha)i}_{(\beta)r} = \binom{A}{B} X^{i(A-B)}_{r} \qquad A=\alpha \text{ and } B=\beta$$

where

$$(1.\ 2) \qquad X^{(\alpha)i}_{(\beta)r} = \frac{\partial x^{(\alpha)i}}{\partial x^{(\beta)r}} \qquad X^i_r = \frac{\partial x^i}{\partial x^r} \quad \text{and} \quad x^{(\alpha)i} = \frac{d^\alpha x^i}{dt^\alpha}$$

the fundamental metric extensor and the extended Christoffel symbols were defined respectively, such that [2]

$$(1.\ 3) \qquad \begin{aligned} g_{\alpha i \beta j} &= \binom{M}{AB} g_{ij}{}^{(M-A-B)} && M \geq A+B, \quad A=\alpha \quad \text{and} \quad B=\beta \\ g^{\alpha i \beta j} &= \begin{bmatrix} AB \\ M \end{bmatrix} g^{ij(A+B-M)} && A+B \geq M \\ \Gamma^{\gamma k}_{\alpha i \beta j} &= \binom{C}{AB} \Gamma^{k}_{ij}{}^{(C-A-B)} && C \geq A+B \quad \text{and} \quad C=\gamma \end{aligned}$$

where

$$\begin{aligned} \binom{A}{BC} &= \frac{A!}{B!\,C!\,(A-B-C)!} && A \geq B+C \\ &= 0. && A < B+C \\ \begin{bmatrix} AB \\ M \end{bmatrix} &= \frac{A!\,B!}{M!\,(A+B-M)!} && A+B \geq M \\ &= 0. && A+B < M \end{aligned}$$

Here, we define the curvature extensor $R^{\alpha i}_{\beta j\gamma k\delta l}$ corresponding to the curvature tensor R^{i}_{jkl} in V_n, such that

(1. 4) $\quad R^{\alpha i}_{\beta j\gamma k\delta l}=\binom{A}{BCD}R^{i}_{jkl}{}^{(A-B-C-D)}\quad A=\alpha,\ B=\beta,\ C=\gamma$ and $D=\delta$

where

$$\binom{A}{BCD}=\frac{A!}{B!\,C!\,D!\,(A-B-C-D)!}\qquad A\geq B+C+D$$
$$=0\qquad\qquad A<B+C+D$$

therefore, we can easily calculate by using (1. 1)

(1. 5) $\quad R^{\alpha i}_{\beta j\gamma k\delta l}=\Gamma^{\alpha i}_{\beta j\gamma k,\,\delta l}-\Gamma^{\alpha i}_{\beta j\delta l,\,\gamma k}+\Gamma^{\varepsilon s}_{\beta j\gamma k}\Gamma^{\alpha i}_{\varepsilon s\delta l}-\Gamma^{\varepsilon s}_{\beta j\delta l}\Gamma^{\alpha l}_{\varepsilon s\gamma k}$

and for any exvector $\xi^{\alpha i}$, or $\xi_{\beta j}$

(1. 6)
$$\xi^{\alpha i}{}_{;\gamma k;\delta l}-\xi^{\alpha i}{}_{;\delta l;\gamma k}=\xi^{\beta j}R^{\alpha i}_{\beta j\gamma k\delta l}$$
$$\xi_{\beta j;\gamma k;\delta l}-\xi_{\beta j;\delta l;\gamma k}=-\xi_{\alpha i}R^{\alpha i}_{\beta j\gamma k\delta l}$$

where ';' denotes the excovariant derivative with respect to $\Gamma^{\alpha i}_{\beta j\gamma k}$ [3].

2. Bochner's lemma on the extensor field.

In the above Riemannian manifold V_n, the Laplacean of $\phi(x)$ is defined by

(2. 1) $\quad \Delta\phi=g^{jk}\phi_{:j:k}=g^{jk}\dfrac{\partial^2\phi}{\partial x^j\partial x^k}-g^{jk}\left\{{i\atop jk}\right\}\dfrac{\partial\phi}{\partial x^i}$

where ':' denotes the covariant derivative with respect to Γ^{k}_{ij}

Here, we dfine the ex-Laplacean of $\Psi(x(t))=\phi^{(M)}(x,t))$ such that

(2. 2) $\quad \overline{\triangle}\Psi=g^{\beta j\gamma k}\phi^{(M)}{}_{;\beta j;\gamma k}=g^{\beta j\gamma k}\dfrac{\partial^2\phi^{(M)}}{\partial x^{(\beta)j}\partial x^{(\gamma)k}}-g^{\beta j\gamma k}\Gamma^{\alpha i}_{\beta j\gamma k}\dfrac{\partial\phi^{(M)}}{\partial x^{(\alpha)i}}$

then, we can easily see the following relation by using (1. 1) and (1. 3)

(2. 3) $\quad \overline{\triangle}\Psi=(M+1)\Delta\phi\qquad (M\geq 0)$

therefore, if a exfunction Ψ satisfies $\overline{\triangle}\Psi\geq 0$, then $\Delta\phi\geq 0$ is satisfied for a function $\phi(x)$. We can apply the so-called Bochner's lemma on the above extensor field:

THEOREM 2. 1 *In a compact Riemannian manifold with positive definite metric, if a exfunction* $\Psi(x(t))=\phi^{(M)}(x(t))$, *satisfies*

$$\overline{\triangle}\phi^{(M)}\geq 0\qquad i.e.\qquad \overline{\triangle}\Psi\geq 0\qquad (M\geq 0)$$

everywhere in the manifold, then we have

$$\Psi=0,\quad \phi=constant\quad and\quad \overline{\triangle}\Psi=0\qquad (M\geq 0)$$

everywhere in the manifold. [1]

Theorem 2.5 in [1] implies as follows

THEOREM 2.2 *In a compact orientable Riemannian manifold* V_n, *for any exscalar field* $\Psi(x(t))=\phi^{(M)}(x(t))$, *we have*

(2.4) $$\int_{V_n}\overline{\triangle}\phi^{(M)}dV=0 \qquad [1]$$

In this paper we consider only the exvector such that

(2.5) $$\xi^{\alpha i}=\xi^{i(A)} \qquad A=\alpha,$$
$$\xi_{\alpha i}=g_{\alpha i\beta j}\xi^{\beta j}=\binom{M}{A}\xi_i^{(M-A)}$$

and we put

(2.6) $$\Psi=\xi^{\alpha i}\xi_{\alpha i}, \qquad \phi=\xi^i\xi_i$$

then we can easily see by (2.5)

(2.7) $$\Psi=\sum_A\binom{M}{A}\xi^{i(A)}\xi_i^{(M-A)}=(\xi^i\xi_i)^{(M)}=\phi^{(M)}$$

and

(2.8) $$\overline{\triangle}\Psi=(M+1)\triangle\phi$$

On the otherehand, by a straightforward calculation, we find

(2.9) $$\overline{\triangle}\Psi=2(\xi^{\alpha i;\beta j}\xi_{\alpha i;\beta j}+g^{\alpha i\beta j}\xi^{\eta k}\xi_{\eta k;\alpha i;\beta j})$$

where we have put

$$\xi^{\alpha i;\beta j}=\xi^{\alpha i}{}_{;\eta k}g^{\beta j\eta k}$$

Now

$$\xi^{\alpha i;\beta j}\xi_{\alpha i;\beta j}=(M+1)\xi^{i:j}\xi_{i:j}$$

is a positive definite form in $\xi_{i:j}$, and

$$g^{\alpha i\beta j}(\xi^{\eta k}\xi_{\eta k;\alpha i;\beta j})=(M+1)g^{ij}\xi^k\xi_{k:i:j}$$
$$T_{\alpha i\beta j}\xi^{\alpha i}\xi^{\beta j}=(T_{ij}\xi^i\xi^j)^{(M)}$$

therefore if ξ_i satisfies

(2.10) $$g^{ij}\xi^k\xi_{k:i:j}=\frac{1}{M+1}(T_{ij}\xi^i\xi^j)^{(M)}$$

and if the quadratic form $T_{ij}\xi^i\xi^j$ satisfies

$$(T_{ij}\xi^i\xi^j)^{(M)}\geqq 0$$

then we have

$$\overline{\triangle}\Psi\geqq 0$$

Conseqently, from Theorem 2.1, we get

$$\overline{\triangle}\Psi=0, \qquad \Psi=0$$

or

$$\xi_{\alpha i;\beta j}=0$$

and also $T_{\alpha i\beta j}\xi^{\alpha i}\xi^{\beta j}=0$, and if the exquadratic form $T_{\alpha i\beta j}\xi^{\alpha i}\xi^{\beta j}$ is positive definite, then we can conclude from $T_{\alpha i\beta j}\xi^{\alpha i}\xi^{\beta j}=0$ that $\xi^{\alpha i}=0$. Thus we have

THEOREM 2. 3 *In a compact Riemannian manifold V_n, there exists no exvector $\xi^{\alpha i}=\xi^{i(A)}$ which satisfies relations*

$$g^{ij}\xi^k\xi_{k:i:j}=\frac{1}{M+1}(T_{ij}\xi^i\xi^j)^{(M)}, \qquad (T_{ij}\xi^i\xi^j)^{(M)}\geq 0$$

unless we have

$$\xi_{\alpha i;\beta j}=0$$

and then automatically $T_{\alpha i\beta j}\xi^{\alpha i}\xi^{\beta j}=0.$

3. Harmonic exvectors and Killing exvectors.

An exvector is called harmonic exvector if it satisfies the conditions

$$(3.\ 1)\qquad \begin{aligned}&\xi_{\alpha i;\beta j}-\xi_{\beta j;\alpha i}=0\\ &\xi^{\alpha i}{}_{;\alpha i}=(M+1)\xi^i{}_{:i}=0\end{aligned}$$

and Killing exvector if

$$(3.\ 2)\qquad \xi_{\alpha i;\beta j}+\xi_{\beta j;\alpha i}=0$$

We can easily see that

$$(3.\ 3)\qquad \begin{aligned}&\xi_{\alpha i;\beta j}\pm\xi_{\beta j;\alpha i}=\tbinom{M}{AB}[\xi_{i:j}\pm\xi_{j:i}]^{(M-A-B)}\\ &\qquad A=\alpha \quad\text{and}\quad B=\beta\end{aligned}$$

From (1. 6) we obtain

$$(3.\ 4)\qquad \xi_{\delta l;\beta j;\gamma k}-(\xi_{\delta l;\beta j}-\xi_{\beta j;\delta l})_{;\gamma k}-\xi_{\beta j;\gamma k;\delta l}=-\xi_{\alpha i}R^{\alpha i}{}_{\beta j\delta l\gamma k}$$

or, multiplying by $g^{Mt\delta l}$ and contracting

$$\begin{aligned}\xi^{Mt}{}_{;\beta j;\gamma k}-g^{Mt\delta l}[\xi_{\delta l;\beta j}-\xi_{\beta j;\delta l}]_{;\gamma k}-g^{Mt\delta l}\xi_{\beta j;\gamma k;\delta l}\\ =-g^{Mt\delta l}\xi_{\alpha i}R^{\alpha i}{}_{\beta j\delta l\gamma k}\end{aligned}$$

Futhere-more, we can easily see that

$$\begin{aligned}g^{\beta j\gamma k}g^{Mt\delta l}\xi_{\beta j;\gamma k;\delta l}&=\sum_{B,C}\left[{}^{BC}_{M}\right]g^{jk(B+C-M)}\tbinom{M}{BC}(g^{tl}\xi_{j:k:l})^{(M-B-C)}\\ &=(M+1)g_{jk}g^{tl}\xi_{j:k:l}=(M+1)g^{tl}\xi^j{}_{:j:l}\end{aligned}$$

Thus, if the exvector $\xi^{\alpha i}$ be harmonic then it satisfies

$$(3.\ 5)\qquad g^{\beta j\gamma k}\xi^{Mt}{}_{;\beta j;\gamma k}=-g^{\beta j\gamma k}g^{Mt\delta l}\xi_{\alpha i}R^{\alpha i}{}_{\beta j\delta l\gamma k}$$

Conversely if (3. 5) be satisfied, by the relations

$$\begin{aligned}&g^{\beta j\gamma k}\xi^{Mt}{}_{;\beta j;\gamma k}=(M+1)g^{jk}\xi^t{}_{:j:k}\\ &-g^{\beta j\gamma k}g^{Mt\delta l}\xi_{\alpha i}R^{\alpha i}{}_{\beta j\delta l\gamma k}=(M+1)R^t{}_j\xi^j\end{aligned}$$

we have

$$g^{jk}\xi^{t}{}_{;j;k}=R^{t}{}_{j}\xi^{j}$$

then, ξ^{i} be harmonic vector, ([1], Theorem 2. 15) therefore by (3. 3) $\xi^{\alpha i}$ also be harmonic exvector.

As to the Killing exvector we have also the same assertion by using

$$-\xi_{\delta l;\beta j;\gamma k}+(\xi_{\delta l;\beta j}+\xi_{\beta j;\delta l})_{;\gamma k}-\xi_{\beta j;\gamma k;\delta l}=-\xi_{\alpha i}R^{\alpha i}{}_{\beta j\delta l\gamma k} \tag{3. 6}$$

and

$$\begin{aligned} &g^{\beta j\gamma k}\xi^{Mt}{}_{;\beta j;\gamma k}=g^{\beta j\gamma k}g^{Mt\delta l}\xi_{\alpha i}R^{\alpha i}{}_{\beta j\delta l\gamma k}\\ &\xi^{\alpha i}{}_{;\alpha i}=0 \end{aligned} \tag{3. 7}$$

Consequently we have the following:

THEOREM 3. 1 *In a compact orientable Riemannian manifold V_n, a necessary and sufficient condition that an exvector $\xi^{\alpha i}$ which derived from ξ^{i} by* (2. 5) *be a harmonic one or a Killing one is that it satisfies* (3. 5) *and* (3. 7) *respectively.*

Let $\xi_{\alpha i}$ be a harmonic exvector, then from (3. 4) we have

$$(\xi_{\alpha i;\beta j;\gamma k}-\xi_{\beta j;\gamma k;\alpha i})\eta^{\alpha i}=-\xi_{\lambda t}R^{\lambda t}{}_{\beta j\alpha i\gamma k}\eta^{\alpha i}$$

or multiplying by $g^{\beta j\gamma k}$ and contracting

$$\begin{aligned} g^{\beta j\gamma k}(\xi_{\alpha i;\beta j;\gamma k}\eta^{\alpha i})&=-g^{\beta j\gamma k}(\xi_{\lambda t}R^{\lambda t}{}_{\beta j\alpha i\gamma k}\eta^{\alpha i}\\ &=-(M+1)g^{jk}\xi_{t}R^{t}{}_{j;k}\eta^{i}=(M+1)R_{ti}\xi^{t}\eta^{i} \end{aligned} \tag{3. 8}$$

Further-more let $\eta^{\alpha i}$ be a Killing exvector, then from (3. 6) we have

$$\eta^{\alpha i}{}_{;\beta j;\gamma k}+\eta_{\beta j;\gamma k;\delta l}g^{\alpha i\delta l}=(\eta_{\lambda t}R^{\lambda t}{}_{\beta j\delta l\gamma k})g^{\alpha i\delta l}$$

or

$$\xi_{\alpha i}(\eta^{\alpha i}{}_{;\beta j;\gamma k}+\eta_{\beta j;\gamma k;\delta l}g^{\alpha i\delta l})=\xi_{\alpha i}(\eta_{\lambda t}R^{\lambda t}{}_{\beta j\delta l\gamma k}g^{\alpha i\delta l})$$

or multiplying by $g^{\beta j\gamma k}$ and contracting,

$$\begin{aligned} g^{\beta j\gamma k}(\xi_{\alpha i}\eta^{\alpha i}{}_{;\beta j;\gamma k})&=g^{\beta j\gamma k}(\xi_{\alpha i}\eta_{\lambda t}R^{\lambda t}{}_{\beta j\delta l\gamma k}g^{\alpha i\delta l})\\ &=(M+1)g^{jk}\xi_{i}\eta_{t}R^{t}{}_{jlk}g^{il}=-(M+1)R_{ti}\xi^{t}\eta^{i} \end{aligned} \tag{3. 9}$$

Here, let us call $\xi_{\alpha i}\eta^{\alpha i}$ the exlinear product of $\xi_{\alpha i}$ and $\eta^{\alpha i}$. If we apply the operator $\overline{\triangle}$ to the exlinear product of these two exvectors, we obtain

$$\begin{aligned} \overline{\triangle}(\xi_{\alpha i}\eta^{\alpha i})=g^{\beta j\gamma k}(\xi_{\alpha i;\beta j;\gamma k}\eta^{\alpha i})+2\xi_{\alpha i;\beta j}\eta^{\alpha i;\beta j}\\ +g^{\beta j\gamma k}(\xi_{\alpha i}\eta^{\alpha i}{}_{;\beta j;\gamma k}) \end{aligned}$$

but, on the other hand, we have

$$\xi_{\alpha i;\beta j}\eta^{\alpha i;\beta j}=0$$

then, from (3. 8) and (3. 9) we get

$$\overline{\triangle}(\xi_{\alpha i}\eta^{\alpha i})=0$$

Therefore, by Theorem 2. 1,

$$\xi_{\alpha i}\eta^{\alpha i}=0$$

and consequently we have the following

THEOREM 3. 2 *In a compact Riemannian manifold V_n, the exlinear product of a harmonic exvector and a Killing exvector is zero.* [1]

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