ON THE SEMI-SIMPLE GROUP SPACE WITH A KAEHLERIAN METRIC

By Sang-Seup Eum

1.

Let V be an orientable manifold of C^{ω}-class, of $2n$ dimensions. We consider a set of transformations of V, which are in one to one correspondence with the points of a space M, and we confine ourselves to the case in which M is a manifold of $2r$ dimensions $(r \geq n)$. Furthermore we assume that the set of transformations form an $2r$-parameter compact semi-simple group, and that the following conditions are satisfied. Let $(u_1, \cdots, u_{2r})$ be a local coordinate system on N valid in some neighbourhood, and let A be any point of this neighbourhood. The transformations T_A transform the points of a neighbourhood N of V into points of a neighbourhood N' of V. If $(x_1, \cdots, x_{2n})$ is a coordinate system valid in N, and $(x'_1, \cdots, x'_{2n})$ is a coordinate system valid in N', the transformation T_A where A has coordinates $(u_1, \cdots, u_{2r})$, transformes the point P, whose coordinates are $(x_1, \cdots, x_{2n})$ into the point P' whose coordinates $(x'_1, \cdots, x'_{2n})$ be given by

$$x'_i = \varphi_i(x_1, \cdots, x_{2n}; u_1, \cdots, u_{2r}) \tag{1. 1}$$

the functions are real analytic functions of $(x_1, \cdots, x_{2n})$ and of $(u_1, \cdots, u_{2r})$, and the determinant $|\partial\varphi^i/\partial x_j|$ different from zero at any point of N for all positions of A. [1]

Now, if we put $\bar{\alpha} = n + \alpha$, $\bar{i} = r + i$ and

$$z_\alpha = x_\alpha + \sqrt{-1}x_{\bar{\alpha}}, \quad \bar{z}_\alpha = x_\alpha - \sqrt{-1}x_{\bar{\alpha}} \qquad (\alpha = 1, \cdots, n)$$
$$s_i = u_i + \sqrt{-1}u_{\bar{i}}, \quad \bar{s}_i = u_i - \sqrt{-1}u_{\bar{i}} \qquad (i = 1, \cdots, r)$$

then we get following relations instead of (1. 1)

$$z'_X = \varphi_X(z_\alpha, \bar{z}_\alpha; s_i, \bar{s}_i) \qquad (X = 1, \cdots, n, \bar{1}, \cdots, \bar{n}) \tag{1. 2}$$

If we eliminate $(z_1, \cdots, z_n, \bar{z}_1, \cdots, \bar{z}_n)$ from equations (1. 2) and the equations

$$\frac{\partial z'_X}{\partial s_I} = \frac{\partial}{\partial s_I}\varphi_X(z, \bar{z}; s, \bar{s}) \qquad (I = 1, \cdots, r, \bar{1}, \cdots, \bar{r}).$$

we obtain

(1. 3) $$\frac{\partial z'_X}{\partial s_I}=\xi^X{}_A(z,\bar z)A^A{}_I(s,\bar s)\ (A,I=1,\cdots,r,\bar 1,\cdots,\bar r)$$

If we apply the conditions of integrability of (1. 3), we obtain the equations

(1. 4) $$\xi^Y{}_A\frac{\partial\xi^X{}_B}{\partial z_Y}-\xi^Y{}_B\frac{\partial\xi^X{}_A}{\partial z_Y}=C^C_{AB}\xi^X{}_C$$

where

(1. 5) $$C^C_{AB}=B_A{}^IB_B{}^J\left(\frac{\partial A^C{}_I}{\partial s_J}-\frac{\partial A^C{}_J}{\partial s_I}\right),\quad \text{where } A^C{}_JB_B{}^J=\delta^C_B$$

and z was used instead of z' ' for convenience.

If we assume that

$$g_{ab}=C^e_{ad}C^d_{eb}=0\qquad (a,\ b,\cdots=1,\cdots,r)\ (conj.)$$

and put(*)

$$g_{a\bar b}=C^e_{a\bar e}C^{\bar e}_{e\bar b}$$

then, for a semi-simple group the rank of the matrix (g_{AB}) is $2n$ and, since the space is compact, the quadratic form $g_{a\bar b}u^a\bar u^b$ is positive definite. Thus, denoting by$(g^{a\bar b})$ the inverse of the matrix $(g_{b\bar e})$, we can use $g^{a\bar b}$ and $g_{b\bar e}$ for raising up and lowing down the indicies.

If we put

$$g^{\alpha\bar\beta}=\xi^\alpha{}_a\xi^{\bar\beta}{}_{\bar b}g^{a\bar b}$$

and

$$\xi_\alpha{}^a=g^{a\bar b}g_{\alpha\bar\beta}\xi^{\bar\beta}{}_{\bar b}$$

where $(g_{\alpha\bar\beta})$ is the inverse matrix of $(g^{\alpha\bar\beta})$, then we have

(1. 6) $$g_{\alpha\bar\beta}=\xi_\alpha{}^a\xi_{\bar\beta}{}^{\bar b}g_{a\bar b},\qquad \xi^\alpha{}_a\xi_\beta{}^a=\delta^\alpha_\beta$$

If $r=n$, we have $\xi^\alpha{}_c\xi_\alpha{}^b=\delta^b_a$,

but if $r>n$, $\xi^\alpha{}_a\xi_\alpha{}^b\neq\delta^b_a$

We assume that the functions $\xi^X{}_A$ are complex analytic in this section, i. e. $\xi^\alpha{}_b$ and $\xi^\alpha{}_{\bar b}$ are functions of z^α only, $\xi^{\bar\alpha}{}_b$ and $\xi^{\bar\alpha}{}_{\bar b}$ are functions of $\bar z^\alpha$ only, then we have

$$g^{\gamma\bar\delta}\frac{\partial g^{\alpha\bar\beta}}{\partial z_\gamma}-g^{\gamma\bar\beta}\frac{\partial g^{\alpha\bar\delta}}{\partial z_\gamma}=\xi^{\bar\delta}{}_{\bar e}\xi^{\bar\beta}{}_{\bar b}g^{c\bar e}g^{a\bar b}\left(\xi^\gamma{}_c\frac{\partial\xi^\alpha{}_a}{\partial z_\gamma}-\xi^\gamma{}_a\frac{\partial\xi^\alpha{}_c}{\partial z_\gamma}\right)$$

but on the other hand from (1. 4) we get

(*) In this paper we assume the self-adjointness on the all indicies

$$\xi^{\beta}{}_{a}\frac{\partial \xi^{\alpha}{}_{b}}{\partial z_{\beta}}-\xi^{\beta}{}_{b}\frac{\partial \xi^{\alpha}{}_{a}}{\partial z_{\beta}}=C^{c}_{ab}\xi^{\alpha}{}_{c}+C^{\bar{c}}_{ab}\xi^{\alpha}{}_{\bar{c}}$$

$$\xi^{\beta}{}_{a}\frac{\partial \xi^{\alpha}{}_{\bar{b}}}{\partial z_{\beta}}-\xi^{\beta}{}_{\bar{b}}\frac{\partial \xi^{\alpha}{}_{a}}{\partial z_{\beta}}=C^{c}_{a\bar{b}}\xi^{\alpha}{}_{c}+C^{\bar{c}}_{a\bar{b}}\xi^{\alpha}{}_{\bar{c}}$$

therefore we obtain

$$g^{\gamma\bar{\delta}}\frac{\partial g^{\alpha\bar{\beta}}}{\partial z_{\gamma}}-g^{\gamma\bar{\beta}}\frac{\partial g^{\alpha\bar{\delta}}}{\partial z_{\gamma}}=\xi^{\bar{\delta}}{}_{\bar{e}}\xi^{\bar{\beta}}{}_{\bar{b}}g^{c\bar{e}}g^{a\bar{b}}(C^{c}_{ab}\xi^{\alpha}{}_{c}+C^{\bar{c}}_{ab}\xi^{\alpha}{}_{\bar{c}})$$

and the following : [3]

THEOREM 1. 1 *When $r=n$, a necessary and sufficient condition that* (1. 6) *is a Kaehlerian metric tensor is $C^{c}_{\bar{a}b}=0$ and $C^{\bar{c}}_{ab}=0$, and when $r>n$ if $C^{c}_{\bar{a}b}=0$ and $C^{\bar{c}}_{ab}=0$ then the metric tensor* (1. 6) *is a Kaehlerian.*

Under the Kaehlerian condition, the Christoffel symbols are given by

$$\Gamma^{\alpha}_{\beta\gamma}=\xi^{\alpha}{}_{a}\frac{\partial \xi_{\beta}{}^{a}}{\partial z_{\gamma}}=-\xi_{\beta}{}^{a}\frac{\partial \xi^{\alpha}{}_{a}}{\partial z_{\gamma}} \qquad when\ r=n$$

$$\Gamma^{\alpha}_{\beta\gamma}=\xi^{\alpha}{}_{a}\xi^{\bar{\varepsilon}}{}_{e}\frac{\partial \xi_{\beta}{}^{b}}{\partial z_{\gamma}}\xi_{\bar{\varepsilon}}{}^{\bar{e}}g^{ae}g_{b\bar{e}} \qquad when\ r>n$$

We consider only the case of $r=n$ in this section, we can easily see that

$$\xi^{\alpha}{}_{a;\gamma}=\frac{\partial \xi^{\alpha}{}_{a}}{\partial z_{\gamma}}+\xi^{\beta}{}_{a}\Gamma^{\alpha}_{\beta\gamma}=0$$

and

$$\xi^{\alpha}{}_{a;\gamma;\bar{\delta}}-\xi^{\alpha}{}_{a;\bar{\delta};\gamma}=\xi^{\beta}{}_{a}R^{\alpha}{}_{\beta\gamma\bar{\delta}}=0$$

where ; indicates the covariant derivative w.r.t. $\Gamma^{\alpha}_{\beta\gamma}$ and $R^{\alpha}{}_{\beta\gamma\bar{\delta}}$ is the curvature tensor constructed by $\Gamma^{\alpha}_{\beta\gamma}$

Let $z^{\alpha}=z^{\alpha}(s)$ is a curve in V, and put

$$\frac{dz^{\alpha}}{ds}=e^{a}\xi^{\alpha}{}_{a}$$

where e^{a} are constants, then we can obtain [2]

$$\frac{d^2z^{\alpha}}{ds^2}+\Gamma^{\alpha}_{\beta\gamma}\frac{dz^{\beta}}{ds}\frac{dz^{\gamma}}{ds}=0 \tag{1. 7}$$

and we shall call (1. 7) is the equation of geodesic. Hence we have the following :

THEOREM 1. 2 *Under the assumption that $\xi^{x}{}_{A}$ are complex analytic functions of z, and that $r=n$, if metric tensor* (1. 6) *satisfies the Kaehlerian condition then the following properties are satisfied.*

(i) $\xi^{\alpha}{}_{a}$ *be parallel,*

(ii) V *is a flat Kaehlerian manifold,* $(R^{\alpha}{}_{\beta\gamma\bar{\delta}}=0)$

(iii) *Curve* $z^{\alpha}=z^{\alpha}(s)$ *whose tangential vector is* $e^{a}\xi^{\alpha}{}_{a}$ *is geodesic, where* e^{a} *are constants.*

If we define $g_{i\bar{j}}$ on M by the relations

$$g_{i\bar{j}}=\sum_{c}A^{c}{}_{i}A^{c}{}_{\bar{j}} \tag{1. 8}$$

and further assume that $A^{C}{}_{I}$ also are complex analytic, i. e. $A^{c}{}_{i}$ and $A^{\bar{c}}{}_{i}$ are functions of s_i only, $A^{c}{}_{\bar{i}}$ and $A^{\bar{c}}{}_{\bar{i}}$ are functions of $\bar{s}_i$ only, then we obtain

$$\frac{\partial g_{i\bar{j}}}{\partial s_k}-\frac{\partial g_{k\bar{j}}}{\partial s_i}=A^{c}{}_{\bar{j}}\left(\frac{\partial A^{c}{}_{i}}{\partial s_k}-\frac{\partial A^{c}{}_{k}}{\partial s_i}\right)$$

$$\frac{\partial g_{i\bar{j}}}{\partial \bar{s}_k}-\frac{\partial g_{i\bar{k}}}{\partial \bar{s}_j}=A^{c}{}_{i}\left(\frac{\partial A^{c}{}_{\bar{j}}}{\partial \bar{s}_k}-\frac{\partial A^{c}{}_{\bar{k}}}{\partial \bar{s}_j}\right)$$

and we have from (1. 5)

$$C^{c}_{ab}=B_a{}^{i}B_b{}^{j}\left(\frac{\partial A^{c}{}_{i}}{\partial s_j}-\frac{\partial A^{c}{}_{j}}{\partial s_i}\right)+B_a{}^{\bar{i}}B_b{}^{\bar{j}}\left(\frac{\partial A^{c}{}_{\bar{i}}}{\partial \bar{s}_j}-\frac{\partial A^{c}{}_{\bar{j}}}{\partial \bar{s}_i}\right)$$

$$C^{\bar{c}}_{ab}=B_a{}^{i}B_b{}^{j}\left(\frac{\partial A^{\bar{c}}{}_{i}}{\partial s_j}-\frac{\partial A^{\bar{c}}{}_{j}}{\partial s_i}\right)+B_a{}^{\bar{i}}B_b{}^{\bar{j}}\left(\frac{\partial A^{\bar{c}}{}_{\bar{i}}}{\partial \bar{s}_j}-\frac{\partial A^{\bar{c}}{}_{\bar{j}}}{\partial \bar{s}_i}\right)$$

therefore if $C^{c}_{ab}=C^{\bar{c}}_{ab}=0$ then the Kaehlerian condition is satisfied, and in this case we can easily see that there exist the functions $\varphi^{c}(s)$, $\psi^{c}(\bar{s})$ satisfying

$$g_{i\bar{j}}=\sum_{c}\frac{\partial\varphi^{c}(s)}{\partial s_i}\frac{\partial\phi^{c}(\bar{s})}{\bar{s}\partial_j}$$

Further-more by putting

$$g^{i\bar{j}}A^{c}{}_{i}=A_{c}{}^{\bar{j}},\qquad g^{k\bar{j}}A^{c}{}_{\bar{j}}=A_{c}{}^{k}$$

we have the following relations

$$g^{i\bar{j}}=\sum_{c}A_{c}{}^{i}A_{c}{}^{\bar{j}}$$

and

$$\Gamma^{i}_{jk}=g^{i\bar{l}}\frac{\partial g_{\bar{l}j}}{\partial s_k}=A_{a}{}^{i}\frac{\partial A^{a}{}_{j}}{\partial s_k}$$

and

$$A^{c}{}_{j:k}=\frac{\partial A^{c}{}_{j}}{\partial s_k}-A^{c}{}_{i}\Gamma^{i}_{jk}=0$$

$$A^{c}{}_{j:k:\bar{l}}-A^{c}{}_{j:\bar{l}:k}=-A^{c}{}_{i}R^{i}{}_{jk\bar{l}}=0$$

where : indicates the covariatant derivative w. r. t. Γ^{i}_{jk} and $R^{i}{}_{jk\bar{l}}$ is the curvature tensor constructed *by* Γ^i_{jk}

Let $s^i = s^i(t)$ is a curve in M, and put

$$\frac{ds^i}{dt} = e^a A_a{}^i$$

then we can obtain

(1. 9) $$\frac{d^2 s^i}{dt^2} + \Gamma^i_{jk} \frac{ds^j}{dt} \frac{ds^k}{dt} = 0$$

and we shall call it is geodesic.

Therefore we may now conclude as follows:

THEOREM 1. 3 *Under the assumption that* $A^c{}_I$ *are complex analytic functions of* s, *if metric tensor* (1. 8) *satisfies the Kaehlerian condition then the following properties are satisfied.*

(i) $A^c{}_i$ *is a parallel gradient vector*

(ii) M *is a flat Kaehlerian manifold.* ($R^i{}_{jk\bar{l}} = 0$)

iii) *Curve* $s^i = s^i(t)$ *whose tangential vector is* $e^a A_a{}^i$ *is geodesic.*

(iv) *Metric tensor of* V *which is defined by* (1. 6) *also satisfies the Kaehlerian condition for all* r *such that* $r \geqq n$, *therefore* V *holds all properties of Theorem* 1. 2.

2.

Take a compact semi-simple group space with Maurer-Cartan equations **

(2. 1) $$h^\beta{}_b \frac{\partial h^\alpha{}_c}{\partial z_\beta} - h^\beta{}_c \frac{\partial h^\alpha{}_b}{\partial z_\beta} = C^a_{bc} h^\alpha{}_a$$

where

(2. 2) $$C^a_{bc} = A_b{}^i A_c{}^j \left(\frac{\partial A^a{}_i}{\partial s_j} - \frac{\partial A^a{}_j}{\partial s_i} \right), \qquad where \qquad A^a{}_i A_a{}^j = \delta^j_i$$

and

$$C^a_{bc} = -C^a_{cb}$$
$$C^e_{ab} C^f_{ce} + C^e_{bc} C^f_{ae} + C^e_{ca} C^f_{be} = 0$$

Now, if we put

$$\frac{1}{2}(x_\alpha + \sqrt{-1} s_\alpha) = z_\alpha, \quad \frac{1}{2}(x_\alpha - \sqrt{-1} s_\alpha) = \bar{z}_\alpha$$

(** In this section we assume that the all indicies take the values 1, 2, ..., n unless otherwise stated.

then

(2. 3) $$x_\alpha = z_\alpha + \bar{z}_\alpha, \qquad s_\alpha = \frac{1}{\sqrt{-1}}(z_\alpha - \bar{z}_\alpha)$$

and $h^\alpha{}_a$ and $A^a{}_i$ are functions of z^α and $\bar{z}^\alpha$, and

$$\frac{\partial h^\alpha{}_a}{\partial x_\gamma} = \frac{\partial h^\alpha{}_a}{\partial z_\gamma}, \qquad \frac{\partial A^a{}_i}{\partial s_\gamma} = \sqrt{-1}\frac{\partial A^a{}_i}{\partial z_\gamma}$$

Here we shall write α, β, γ, $\cdots$ instead of i. j. k, $\cdots$ and put

$$\overline{h^\alpha{}_a(z,\bar{z})} = h^{\bar{\alpha}}{}_{\bar{a}}(z,\bar{z}),\ \overline{A^a{}_\alpha(z,\bar{z})} = A^{\bar{a}}{}_{\bar{\alpha}}(z,\bar{z}),\ (\bar{\alpha}, \bar{a} = \bar{1}, \cdots, \bar{n})$$

then

$$h^X{}_A = (h^\alpha{}_a, 0,\ 0,\ h^{\bar{\alpha}}{}_{\bar{a}}), \qquad A^A{}_X = (A^a{}_\alpha,\ 0,\ 0,\ A^{\bar{a}}{}_{\bar{\alpha}})$$

thus, we may obtain pure contravariant vector $h^X{}_A$($A = 1, \cdots, n, \bar{1}, \cdots, \bar{n}$) and pure covariant vector $A^A{}_X$($A = 1, \cdots, n, \bar{1}, \cdots, \bar{n}$), and we also get the following relations

(2. 4)
$$\frac{\partial h^\alpha{}_a}{\partial \bar{z}_\gamma} = \frac{\partial h^\alpha{}_a}{\partial z_\gamma}, \qquad \frac{\partial h^{\bar{\alpha}}{}_{\bar{a}}}{\partial z_\gamma} = \frac{\partial h^{\bar{\alpha}}{}_{\bar{a}}}{\partial \bar{z}_\gamma}$$
$$\frac{\partial A^a{}_\alpha}{\partial \bar{z}_\gamma} = -\frac{\partial A^a{}_\alpha}{\partial z_\gamma}, \qquad \frac{\partial A^{\bar{a}}{}_{\bar{\alpha}}}{\partial z_\gamma} = -\frac{\partial A^{\bar{a}}{}_{\bar{\alpha}}}{\partial \bar{z}_\gamma}$$

From (2. 1) and (2. 2), we get

$$h^\beta{}_b(z,\bar{z})\frac{\partial h^\alpha{}_c(z,\bar{z})}{\partial z_\beta} - h^\beta{}_c(z,\bar{z})\frac{\partial h^\alpha{}_b(z,\bar{z})}{\partial z_\beta} = C^a_{bc} h^\alpha{}_a(z,\bar{z})\ (conj.)$$

$$C^a_{bc} = \sqrt{-1}A_b{}^\beta(z,\bar{z})A_c{}^\gamma(z,\bar{z})\left(\frac{\partial A^a{}_\beta\ z,\bar{z})}{\partial z_\gamma} - \frac{\partial A^a{}_\gamma(z,\bar{z})}{\partial z_\beta}\right)(conj.)$$

(i) By putting

(2. 10)
$$g_{bc} = -C^f_{be}C^e_{cf}$$
$$b_{\beta\gamma} = h_\beta{}^b h_\gamma{}^c g_{bc}$$

where

$$h_\beta{}^b = g^{bc}g_{\beta\gamma}h^\gamma{}_c$$

we obtain

$$h^\alpha{}_a h_\beta{}^a = \delta^\alpha_\beta \quad and \quad h^\alpha{}_a h_\alpha{}^b = \delta^b_a$$

Further-more by putting

(2. 11) $$\Omega_{\beta\gamma}{}^\alpha = \frac{1}{2}C^a_{bc}h_\beta{}^b h_\gamma{}^c h^\alpha{}_a$$

(2. 12) $$E^\alpha{}_{\beta\gamma} = h^\alpha{}_a \frac{\partial h_\beta{}^a}{\partial z_\gamma}$$

we may obtain the following relations by the same way in [3] (pp. 90–92)

$$_{(b)}\left\{{\alpha \atop \beta\gamma}\right\}=\frac{1}{2}(E^{\alpha}{}_{\beta\gamma}+E^{\alpha}{}_{\gamma\beta})$$

$$\Omega_{\beta\gamma}{}^{\alpha}=\frac{1}{2}(E^{\alpha}{}_{\beta\gamma}-E^{\alpha}{}_{\gamma\beta})$$

$$_{(b)}\ R^{\alpha}{}_{\beta\gamma\delta}=\Omega_{\gamma\delta}{}^{\rho}\Omega_{\rho\beta}{}^{\alpha}$$

$$_{(b)}\ R_{\beta\gamma}=\frac{1}{4}b_{\beta\gamma}$$

where $_{(b)}\left\{{\alpha \atop \beta\gamma}\right\}$ are the Christoffel symbols which are calculate usually from $b_{\beta\gamma}$ and $_{(b)}R^{\alpha}{}_{\beta\gamma\delta}$ is the curvature tensor calculated from $_{(b)}\left\{{\alpha \atop \beta\gamma}\right\}$.

(ii) If we put

(2. 20) $$a_{\beta\gamma}=A^{b}{}_{\beta}A^{c}{}_{\gamma}g_{bc}$$

instead of (2. 10) then by putting

(2. 21)
$$\mathrm{L}^{\alpha}{}_{\beta\gamma}=A_{c}{}^{\alpha}\frac{\partial A^{c}{}_{\beta}}{\partial z_{\gamma}}$$
$$\phi_{\beta\gamma}{}^{\alpha}=\frac{1}{2}(L^{\alpha}{}_{\beta\gamma}-L^{\alpha}{}_{\gamma\beta})$$

we may obtain the following relations by the same way in [3] (pp. 90–92)

(2. 22)
$$_{(a)}\ \left\{{\alpha \atop \beta\gamma}\right\}=\frac{1}{2}(L^{\alpha}{}_{\beta\gamma}+L^{\alpha}{}_{\gamma\beta})$$
$$_{(a)}\ R^{\alpha}{}_{\beta\gamma\delta}=\phi_{\gamma\delta}{}^{\sigma}\phi_{\sigma\beta}{}^{\alpha}$$
$$_{(a)}\ R_{\alpha\beta\gamma\delta}=-\phi_{\alpha\beta\sigma}\phi_{\gamma\delta}{}^{\sigma}$$

where $_{(a)}\left\{{\alpha \atop \beta\gamma}\right\}$ are the Christoffel symbols which are calculated ususlly from $a_{\beta\gamma}$, and $_{(a)}R^{\alpha}{}_{\beta\gamma\delta}$ is the curvature tensor calculated from $_{(a)}\left\{{\alpha \atop \beta\gamma}\right\}$, and

$$\phi_{\alpha\beta\sigma}=a_{\alpha\rho}\phi_{\beta\sigma}{}^{\rho}$$

Further-more from (2. 2), (2. 21) and the last of (2. 22) we may also obtain

$$_{(a)}\ R_{\beta\gamma}=-\frac{1}{4}a_{\beta\gamma}$$

(iii) If we put

(2. 30) $$g_{\alpha\bar{\beta}}=h_{\alpha}{}^{a}h_{\bar{\beta}}{}^{\bar{b}}g_{a\bar{b}}$$

then we may obtain the follows by a straightforward calculation

$$g^{\alpha\bar{\beta}}=h^{\alpha}{}_{a}h^{\bar{\beta}}{}_{\bar{b}}g^{a\bar{b}}$$

$$\frac{\partial g_{\alpha\bar{\beta}}}{\partial z_{\gamma}}=g_{\rho\bar{\beta}}E^{\rho}{}_{\alpha\gamma}+g_{\alpha\bar{\rho}}E^{\bar{\rho}}{}_{\bar{\beta}\gamma}\qquad(E^{\bar{\rho}}{}_{\bar{\beta}\gamma}=\overline{E^{\rho}{}_{\beta\gamma}})$$

and the Kaehlerian condition

$$\frac{\partial g_{\alpha\bar{\beta}}}{\partial z_\gamma}=\frac{\partial g_{\gamma\bar{\beta}}}{\partial z_\alpha}$$

is equivalent to

$$g_{\rho\bar{\beta}}\Omega_{\alpha\gamma}{}^{\rho}=g_{\bar{\rho}[\gamma}E^{\bar{\rho}}{}_{|\bar{\beta}|\,\alpha]}$$

where the right hand members indicate

$$g_{\bar{\rho}\gamma}E^{\bar{\rho}}{}_{\bar{\beta}\alpha}-g_{\bar{\rho}\alpha}E^{\bar{\rho}}{}_{\bar{\beta}\gamma}$$

If the above condition is satisfied then

$$\Gamma^{\alpha}_{\beta\gamma}=g^{\alpha\bar{\varepsilon}}\frac{\partial g_{\beta\bar{\varepsilon}}}{\partial z_\gamma}=E^{\alpha}{}_{\gamma\beta}+g^{\alpha\bar{\varepsilon}}g_{\gamma\bar{\rho}}E^{\bar{\rho}}{}_{\bar{\varepsilon}\beta}$$

and contracting by $\alpha=\gamma$ we get

$$\Gamma^{\alpha}_{\beta\alpha}=E^{\alpha}{}_{\alpha\beta}+\overline{E^{\alpha}{}_{\alpha\beta}}$$

then from (2. 4)and (2. 12)

$$R_{\beta\bar{\gamma}}=-\frac{\partial\Gamma^{\alpha}_{\beta\alpha}}{\partial\bar{z}}=-\left(\frac{\partial}{\partial z_\gamma}E^{\alpha}{}_{\alpha\beta}+\overline{\frac{\partial}{\partial z_\gamma}E^{\alpha}{}_{\alpha\beta}}\right)$$

and we may now conclude as follows:

THEOREM 2 *When we introduce the metric tensor* (2. 30) *in our semi-simple group space endowed with complex coordinates* $(z_\alpha,\ \bar{z}_\alpha)$ *by* (2. 3), *if the Kaehlerian condition is satisfied then the Ricci tensor is real.*

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Mathematical Department,
Liberal Arts and Science College
Kyungpook University

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