# UNIFORM TOPOLOGY ON A GROUP

by Mi-Soo Pae

Let $V_\alpha$ be a neighbourhood of identity $e$ in a topological group $G$, then

(1) $\qquad V_\alpha(p)=\{qp;\ q \in V_\alpha\}$

be a neighborhood of $p \in G$ as for uniform topology of $G$.
In § 1, we shall introduce an uniform topology into an abstract group $G$ directly by (1) regarding $V_\alpha$ as a subset of $G$ and prove the uniform topology is discrete. This argument is performed in the similar way taking the following (2) instead of (1).

(2) $\qquad V_\alpha(p)=\{q;\ q=a^{-1}pa,\ a \in V_\alpha\}.$

And we shall study the necessary and sufficient condition of the uniform topology by (1) to be non-discrete a na the sufficient condition of the uniform topology (2) to be non-discrete in § 2, and finally the sufficient conditions that the group operations are continuous under the uniform topologies by (1) and (2) respectively in § 3.

## § 1. Discrete uniform topology on a group $G$

For $a, x, y \in G$, we define $(x, y) \in G\times G$ such as

(3) $\qquad \exists\ a \in G;\ ax=y.$

Let $\{V_\alpha\}$ be the system of all subsets which contain $e$. And we put

(4) $\qquad \Phi=\{U;\ U\subseteq G\times G,\ \mathrm{E}U_\alpha;\ U_\alpha\subseteq U\}$
where $U_\alpha=\{(x,\ y);\ ax=y,\ a \in V_\alpha\}=\{(x,y);\ V_\alpha\}$,

then $\Phi$ be a filter of $G\times G$. Hence we have

THEOREM 1. *$\Phi$ determines an uniform topology of group $G$.*

PROOF. a) Let $\Delta=\{(x,\ x);\{e\}\}$, then $V_\alpha$ contains $e$ for arbitrary $V_\alpha \in \{V_\alpha\}$, itmeans $U \in \Phi$ implies $U\supseteq\Delta$. b) If $(x,\ y) \in U_\alpha=\{(x,\ y);\ V_\alpha\}$, then

$$(y,\ x) \in U_\alpha^{-1}=\{(y,\ x);\ V_\alpha^{-1},\}$$
where $U_\alpha^{-1}=\{(y,\ x);\ (x,\ y) \in U_\alpha\}$,

from $a^{-1}y=x$. Hence, for arbitrary $U \in \Phi$ there exists $U_\alpha \in \Phi$, such that $U_\alpha \in U$ by (4). From $U_\alpha^- \in \Phi$ we obtain $U_\alpha^- \subseteq U^- \in \Phi$. *c*) For any $U \subseteq \Phi$ there exists

(5) $U_\alpha \subseteq U$ such that $U_\alpha=\{(x, y); V_\alpha\} \in \Phi$.

There exists

$$V_\beta \cdot V_\beta \subseteq V_\alpha;\ V_\beta \in \{V_\alpha\},$$
where $V_\beta \cdot V_\beta=\{ab,\ a \in V_\beta,\ b \in V_\beta\}$,

from $ee= e \in V_\alpha$. We put $W=\{(x, y); V_\beta\}$, then

$$WW \subseteq U,$$
where $WW=\{(x, y);\ \exists z \in G:\ (x, z) \in W, (z, y) \in W\}$.

And the filter $\Phi$ introduces an uniform topology into group $G$.

By $\{e\} \in \{V_\alpha\}$ we obtain the following:

COROLLARY 1. *$\Phi$ dertermines a discrete uniform topology in $G$.*

On a topological group $G$ a neighborhood $V_\alpha$ of $e$ is a subgroup of $G$ containing $e$. Then

COROLLARY 2. *If a topological group $G$ is discrete, then the uniform topology of $G$ by* (1) *be equivalent to the uniform topology in theorem* 1.

Now we consider the pair $(x, y) \in G \times G$ when and only when

(6) $\exists\ a \in G;\ a^{-1}xa=y$.

Similarly as for (4), $\Phi$ determines a discrete uniform topology on a group $G$, in the same way above. We shall call

(7) $V_\alpha(x)=\{y;\ a^{-1}xa=y,\ a \in V_\alpha\}$

a conjugate neighbourhood of $x \in G$.

REMARK. When we define $(x, y)$ as $xy=a$, $a \in V_\alpha$, then $U_\alpha$ does not include $\Delta$ if $V_\alpha$ does not contain $\{x^2;\ x \in G\}$. And we put $(x, y^{-1})$ such as $xy=a$, $a \in V_\alpha$. Then, by $V_\alpha^{-1}=V_\beta$ containing $e$, this uniform topology be equivalent to the uniform topology in theorem 1.

## § 2. Non-discrete uniform topology on a group $G$

We shall introduce non-discrete uniform topology into a group. For this purpose we suppose that $G$ is $T_1$-space and group. A neighbourhood of point $x$ in a topological space is a subset which includes an open set containing point $x$. The totality of neighbourhoods of point $x$ is called the fundamental system of neighbourhoods of point $x$, written by $\mathfrak{N}_x$.

From corollary 2 of theorem 1, we get easily following:

THEOREM 2. *The necessary and sufficient condition of the uniform topology* $\dot{G}$ *by* (1) *to be non-discrete is the topological group* $G$ *to be non-discrete. (Where* $G$ *is* $T_1$*-space.)*

Now we consider the uniform topology by conjugate neighbourhoods.

LEMMA 1. *If there is an uniform topology in a* $T_1$*-space and at the same time group* $G$ *by conjugate neighbourhoods* (7), *then for arbitrary* $V_\alpha \in \mathfrak{N}_e$ *there exists some* $V_\beta \in \mathfrak{N}_e$ *such that*

(8) $$V_\beta^{-1}\cdot V_\beta \subseteq V_\alpha, \text{ where } V_\beta^{-1}=\{x^{-1};\ x \in V_\beta\}.$$

PROOF. Let

$$\{V_\alpha(p);\ \alpha \in \Gamma\},$$
$$\text{where } V_\alpha(p)=\{q;\ q=a^{-1}pa,\ a \in V_\alpha\},$$
$$V_\alpha \in \mathfrak{N}_e,$$

be a base of the fundamental system of conjugate neighbourhoods of $p \in G$ in the uniform topology. By the definition of the uniform topology, for arbitrary $\alpha \in \Gamma$ there exists some $\beta \in \Gamma$ such that

$q \in V_\alpha(p)$ if $p \in V_\beta(r)$ and $a \in V_\beta(r)$ for any $r \in G$.

From $p \in V_\beta(r)$ and $q \in V_\beta(r)$, there exist $a$, $b \in V_\beta$ such that $p=a^{-1}ra$, $q=b^{-1}rb$ and $q=(a^{-1}b)^{-1}p(a^{-1}b) \in V_\alpha(p)$. Hence, $V_\beta^{-1}\cdot V_\beta \in V_\alpha$.

From now on in this paper, we assume that $G$ is a $T_1$-space as well as a topo logical group. Then $G$ also satisfies the result of lemma 1.

THEOREM 3.

(A) *For each* $x_1, x_2 \in G$ $(x_1 \neq x_2)$ *there is at least one system* $\{V_x; x \in M\}$ *of neighbourhoods of* $e \in G$ *such that*

(9) $$\cap_{x\in M} V_x = \text{open set and } V_x \not\ni x,$$
$$\text{where } M=M(x_1,\ x_2)=\{a;\ x_2=a^{-1}x_1a,\ a \in G\}.$$

*If* $G$ *satisfies the condition* (A), *then* $G$ *be an uniform space by conjugate neigh-*

***bourhoods.***

PROOF. a) Since $V_\alpha$ for any $\alpha \in \Gamma$ contains $e$, $V_\alpha(x)$ contains point $x$ for arbitrary $x \in G$. b) From (A) there exists at least one $\{V_x\}$ satisfying (9). We put

$$\cap_{x \in M} V_x = V.$$

Since $V$ is open set with $e$, $V$ is a neighbourhood of $e$. Hence there is an $\alpha \in \Gamma$ which $V_\alpha = V$. And $V_\alpha(x_1) \not\ni x_2$. For if $y = a^{-1}x_1a$ and $a \in V_\alpha$ since $a$ does not belong to $M(x_1, x_2)$, then $y$ is different from $x_2$ for any $a \in V_\alpha$. c) For arbitrary $\alpha, \beta \in \Gamma$ there exists some $\gamma \in \Gamma$ such that $V_\gamma \subseteq V_\alpha \cap V_\beta; V_\alpha, V_\beta, V_\gamma \in \mathscr{B}_e$. And we have $V_\alpha(x) \cap V_\beta(x) \supseteq V_\gamma(x)$ for arbitrary $x \in G$. d) For arbitrary $V_\alpha \in \mathscr{B}_e$ there exists $V_\beta$ in $\mathscr{B}_e$, satisfying (8), because $G$ is a topological group. And, if we put $p \in V_\beta(r)$ and $q \in V_\beta(r)$ for arbitrary $r \in G$, then we get $q \in V_\beta(p)$ by similar way on lemma 1.
Here $a^{-1}ea = e$ for any $a \in G$, and have $V_\alpha(e) = \{e\}$ for arbitrary $V_\alpha \in \mathscr{B}_e$ then

COROLLARY 1. *The identity e of G is an isolate point the uniform topology in theorem* 3.

COROLLARY 2. (A′) *There exists $V_\alpha \in \mathscr{B}_e$ such as $V_\alpha \cap M(x_1, x_2)$ is finite set for each $x_1, x_2 \in G$ $(x_1 \neq x_2)$.*

*If G satisfies the (A′), G is an uniform space by conjugate neighbourhoods.*

PROOF. Let's put

$$V_\alpha \cap M(x_1, x_2) = \{a_1, a_2, \cdots\cdots, a_n\}.$$

Then there are $V_i \in \mathscr{B}_e$ such that $V_i \not\ni a_i$ $(i = 1, 2, \cdots\cdots, n)$, for $G$ is a $T_1$-space and $a_i \neq e$ $(i = 1, 2, \cdots, n)$. There exists $V_\beta \in \mathscr{B}_e$ which $V_\beta \subseteq V_\alpha \cap (\cap_{i=1}^n V_i)$ because the $V_\alpha$ and $V_i$'s are the finite number of neighbourhoods of $e$. And we obtain $V_\beta(x_1) \not\ni x_2$.

LEMMA 2. If topological group $G$ satisfies

(B) There exists $V_x \in \mathscr{B}_e$ that $V_x \cap N(x) = \{e\}$ for each $x \in G$,

and,

(C) $\{e\} \notin \mathscr{B}_e$, i.e. $G$ is non-discrete, then $N(x) = \{a;\ ax = xa,\ a \in G\} \notin \mathscr{B}_e$.

PROOF. If it is not so, then there exists some $V_\beta \in \mathscr{B}_e$ such that $V_\beta \subseteq V_\alpha \cap N(x)$ for any $V_\alpha \in \mathscr{B}_e$. From the condition (B), we obtain $\{e\} \in \mathscr{B}_e$ which is contrary to (C).

LEMMA 3. *G satisfies either* (A), (B) *and* (C), *or* (A′), (B) *and* (C), *then there is no neighbourhood* $V_{\alpha}(x)$ *of* $x$ *such that* $V_{\alpha}(x)=\{x\}$ *for each* $x \in G$ $(x \neq e)$.

PROOF. We suppose that there exists $V_{\alpha}(x)$ such that $V_{\alpha}(x)=\{x\}$. Since $\{e\}$ does not belong to $\mathfrak{B}_e$ from (C) there exists point $a$, not identity, which is contained to $V_{\alpha}$. Since $a^{-1}xa=x$ for arbitrary $a \in V_{\alpha}$, then $a$ belongs to $N(x)$. So $V_{\alpha}$ be included in $N(x)$. Since $V_{\alpha} \in \mathfrak{B}_e$, $N(x)$ belongs to $\mathfrak{B}_e$. This is contrary to lemma 2.

Hence we obtain following theorem:

THEOREM 4. *G satisfies either* (A), (B) *and* (C), *or* (A′), (B) *and* (C), *then there exists a non-discrete uniform topology in G, and the identity e is only one isolate point in G.*

### § 3. Uniform topological groups

In general mappings $f(x, y)=xy$ and $\phi(x)=x^{-1}$ are not continuousin uniform topology of $G$ which is introduced from $G$. And particularly we shall call $G$ an uniform topological group when and only when

(a) $G$ is a group,

(b) $G$ is an uniform space,

(c) $f(x, y)=xy$ and $\phi(x)=x^{-1}$ are continuous mappings for the uniform topology of $G$.

THEOREM 5. *The sufficient condition that G is an uniform topological group in the sense of* (D) *is the following* (D):

(D) *For each* $x \in G$ *there exists at least one* $V_x$ *such that*

$$V_x \subseteq N(x), \quad V_x \in \mathfrak{B}_e. \tag{10}$$

PROOF. There are $V_{\beta}$ and $V_{\gamma_1}$ in $\mathfrak{B}_e$ which $V_{\beta}\cdot V_{\gamma_1} \subseteq V_{\alpha}$ for and $V_{\alpha} \in \mathfrak{B}_e$. While there is $V_x$ satisfying (10) for each $x \in G$ from (D). And there is $V_x \in \mathfrak{B}_e$ such that $V_{\gamma} \subseteq V_{\gamma_1} \cap V_x$. If $xy=z$, $x' \in V_{\beta}(x)$ and $y' \in V_{\gamma}(y)$, then $x'y'=(ax)(by)=a(bx)y \subset V_{\alpha}(z)$, that is $V_{\beta}(x)\cdot V_{\gamma}(y) \subseteq V_{\alpha}(z)$. Hence $f(x, y)=xy$ is a continuous mapping from $G \times G$ to $G$. As $V_x$ depends on $x$, so this mapping is not uniformly continuous. Now we shall prove $\phi(x)=x^{-1}$ is also continuous mapping. Let $V(x^{-1})$ be an arbitrary neighbourhood of $x^{-1}$. And there is some $\alpha \in \Gamma$ such that $V_{\alpha}(x^{-1})=\{y;\ ax^{-1}=y,\ a \in V_{\alpha}\} \subseteq V(x^{-1})$. Furthermore there exists $V_{\beta_1} \in \mathfrak{B}_e$ such that $V_{\beta}^{-1} \subseteq V_{\alpha}$ for any $V_\alpha \in \mathscr{B}$. From (D), we put $V_\beta \subseteq V_{\beta_1} \cap V_\gamma$, $V_\beta \in \mathscr{B}_e$, then $x'^{-1}=(ax)^{-1}=(xa)^{-1}=a^{-1}x^{-1} \in V_\alpha(x^{-1})$, $x' \in V_\beta(x)$. Hence we obtain $V_\beta(x)^{-1} \subseteq V_\alpha(x^{-1})$, where $V_\beta(x)^{-1}=\{y^{-1};\ y \in V_\beta(x)\}$.

LEMMA 4. *If G is the uniform space by conjugate neighbourhoods, then* $\phi(x)=x^{-1}$ *is continuous on G.*

PROOF. We have easily $V_\alpha(x)^{-1}=V_\alpha(x^{-1})$ for each $x \in G$.

THEOREM 6.

(E) *For each* $x \in G$, *there exist* $U, V \in \mathscr{B}_e$ *satisfying*

(11) $$U \cdot V \subseteq N(x).$$

*If G satisfies* (A) *and* (E), *then G is an uniform topological group, by conjugate neighbourhoods.*

PROOF. There exist $V_{\beta_1}, V_{\gamma_1} \in \mathscr{B}_e$ such that $V_{\beta_1} \cdot V_{\gamma_1} \subseteq V_\alpha$ for arbitrary $V_\alpha \in \mathscr{B}_e$. From (E), we obtain $V_\beta, V_\gamma$ such that $V_\beta \subseteq V_{\beta_1} \cap U$, and $V_\gamma \subseteq V_{\gamma_1} \cap V$. Then $V_\beta(x) \cdot V_\gamma(y) \subseteq V_\alpha(z)$. By theorem 3 and lemma 4, $G$ is an uniform topological group.

Finally, I express my hearty thanks to professor Chung-Ki Pahk in Kyungpook University for his kind guidance.

Sep. 20, 1957
Mathematical Department
Liberal Arts and Science College
Kyungpook University

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