NOTES ON THE LATTICE ORDERED GROUPS

by Tae-Ho Choe

Introduction. A set L of the elements is called a lattice-ordered group, if L forms a group and lattice at the same time, and in L is order $x \geqq y$, preserved under the multiplication:

$$x \geqq y \text{ implies } xz \geqq yz \text{ and } zx \geqq zy \text{ for all } z \text{ in } L.$$

They have been discussed mainly by G. Birkhoff [1], and T. Nakayama [3], and they have already proved the followings:

(1) A lattice-ordered group is a distributive lattice as a lattice

(2) The order of any element of lattice-ordered group except for the identity is zero.

(3) There is no greatest and least elements in a lattice-ordered group etc.

In this paper, we shall consider the relation between an infinite cyclic group or, in general, a free abelian group and a lattice-ordered group in Theorem 1, 2, and 3, secondly under the interval topology [Definition 1] introduced to a lattice-ordered group L, L is always a discrete topological space or the topological space or the topological space which is composed only of the accumulating points of L in Theorem 4, and finally, prove that if there is no element in a lattice-ordered group which covers the identity then it can be a topological lattice-ordered group [Definition] 2 in Theorem 5.

We recollect some notations, which will be used in this paper.

(A) The notations $A \cup B$, $A \cap B$ for the subsets A, B of lattice-ordered group L means that $\{a \cup b \mid a \in A, b \in B\}$, $\{a \cap b \mid a \in A, b \in B\}$ respectively.

(B) The notations $A \vee B$, $A \wedge B$ means that set-union, set-intersection of A and B respectively.

(C) The notation $a \lessapprox b$ means that the element b covers the element a.

(D) The notation $a \# b$ means that the elements a, b are incomparable, i.e., neither $a \leqq b$ nor $b < a$.

The other notations are the same in [2].

1. Free abelian groups and lattice-ordered groups

LEMMA 1. *If for some element a of a lattice-ordered group L there exist the* *element* x_i $(i=1, 2, \cdots\cdots, n)$ *which* $x_i \gtrsim^{*} a$ *for any* i *and* $y \geqq x_i$ *for some* i *and any element* y *such that* $y>a$ *in* L, *then for arbitrary element* b *of* L, *we have* $x_i a^{-1} b \gtrsim^{*} b$ *for any* i *and* $z \geqq x_i a^{-1} b$ *for some* i *and any element* z *such that* $z>b$ *in* L.

PROOF $x_i \gtrsim^{*} a$ follows $x_i a^{-1} \gtrsim^{*} e$, for, $x_i a^{-1} > p > e$ implies $x_i > pa > a$. And $x_i a^{-1} b \gtrsim^{*} b$ for any i. Since $z>b$ implies $zb^{-1}a>a$. there exists some x_i such that $zb^{-1}a \geqq x_i$, i.e., $z \geqq x_i a^{-1} b$, which completes the proof.

Accordingly, in particular, we have the result of Lemma 1 for $b=e$, the idenity of L, which will be used in the following Theorems.

THEOREM 1 *If for some element* a *of a lattice-ordered group* L *there exists* x *which* $x \gtrsim^{*} a$ *and* $y \geqq x$ *for any* y *such that* $y>a$ *in* L, *then* L *is a chain as a lattice and infinite cyclic group as a group, and vice versa.*

PROOF. By Lemma 1, there exists the element p which $p \gtrsim^{*} e$ and $y \geqq px$ for any $x \in Z$ and any y such that $y>x$ in L. We shall show now that L is a chain. If there exist the elements x, y that $x \# y$ in L, then $p(x \cap y) \gtrsim^{*} x \cap y$, $x > x \cap y$ and $y > x \cap y$, hence $x \geqq p(x \cap y)$ and $y \geqq p(x \cap y)$. And $e \geqq p$, which is contrary.

Next, we shall show that L is an infinite cyclic group. From $e \gtrsim^{*} p$ it is clear $p^{-1} \gtrsim^{*} (p^{-1})^2 = p^{-2}$, and hence we have

$$p^2 \gtrsim^{*} p \gtrsim^{*} e \gtrsim^{*} p^{-1} \gtrsim^{*} p^{-2}.$$

Let x be an arbitrary element of L, then either $x \geqq e$ or $x<e$. We shall consider the case of $x \geqq e$, the another case $x<e$ by the same way. It is easily seen that there existe the subchain $\{x_\alpha\}$ of L such that $e = x_\alpha \gtrsim^{*} x_\beta \gtrsim^{*} \cdots\cdots \gtrsim^{*} x_\lambda = x$, and that $\{x_\alpha\}$ is a well-ordered.

Suppose x_α is expressed by a integer power of p for any x_α such that $x_\alpha < x$ in L. Since $xp^{-1} < x$, we see $xp^{-1} = p^m$ for some integer m, i.e. $x = p^{m+1}$, hence, by induction, all elements of L are expressed by a integer power of p.

And order of any element except for the identity is zero [3].

Conversly, we can easily seen that given infinite cyclic group $L=\{g\}$ be a lattice-ordered group which satisfies the hypotheses of Theorem, if one defines $g^m \geqq g^n$ to mean $m \geqq n$ as integers.

Moreover, we have the following theorem.

THEOREM 2. *If for the identity* e *of a lattice-ordered group* L *there exist the elements* p, q *of* L *such that* $x \geqq p$ *or* $x \geqq q$ *for any* $x>e$ *in* L *and* $p \gtrsim^{*} e, q \gtrsim^{*} e$, *then* L *is a free abelian group with two generators* p, q *and in* L $p^m \cdot q^n \geqq p^a \cdot q^b$ *if and* *only if* $m \geqq a$ *and* $n \geqq b$ *as integers, where* $p \mp q$. *And vice versa.*

PROOF. p, q are commutative. In fact, from $e \mathrel{\overset{*}{\lessgtr}} p$, $e \mathrel{\overset{*}{\lessgtr}} q^{-1}pq$ we see $q^{-1}pq=p$ or $q^{-1}pq=q$. If $q^{-1}pq=q$ then $p=q$ which is contrary. Hence $q^{-1}pq=p$.

We have $p^{-1} \mathrel{\overset{*}{\lessgtr}} e, q^{-1} \mathrel{\overset{*}{\lessgtr}} ep \mathrel{\overset{*}{\lessgtr}} p^2$ and $p \mathrel{\overset{*}{\lessgtr}} pq$ from $p\,q \mathrel{\overset{*}{\lessgtr}} e$. Similarly $q \mathrel{\overset{*}{\lessgtr}} pq, p^{-1} \mathrel{\overset{*}{\lessgtr}} p^{-1}q$, $q^{-1} \mathrel{\overset{*}{\lessgtr}} pq^{-1}$ and $q^{-2} \mathrel{\overset{*}{\lessgtr}} q^{-1}$ etc. In general, $x=p^a q^b$ for arbitrary element x of L which $x \geqq e$ where a, b are integers. In fact, there exists a sub-chain $\{x_\alpha\}$ of L such that

$$e=x_\alpha \mathrel{\overset{*}{\lessgtr}} x_\beta \mathrel{\overset{*}{\lessgtr}} \cdots\cdots\cdots \mathrel{\overset{*}{\lessgtr}} x_\lambda = x.$$

Since for an arbitrary element x_λ of $\{x_\alpha\}$ we can fined the elements x_μ, x_σ such that $x_\mu \mathrel{\overset{*}{\lessgtr}} x_\nu$ $x_\lambda \mathrel{\overset{*}{\lessgtr}} x_\nu$, respectively, and $\{x_\alpha\}$ is well-ordered.

Suppose x_α is expressed by the form $p^a p^b$ for any x_α such that $x_\alpha < x$ in $\{x_\alpha\}$. We get $xp^{-1} \in \{x_a\}$ or $xq^{-1} \in \{x_a\}$ because $xp^{-1} \mathrel{\overset{*}{\lessgtr}} x$ and $xq^{-1} \mathrel{\overset{*}{\lessgtr}} x$, and $xp^{-1}=p^m q^n$ or $xq^{-1}=p^m q^n$, hence $x=p^a q^b$.

Now, let y be arbitrary element of L, and we consider the element $y \cup e$, then we see $y \cup e = p^m q^n \geqq y$ from $e \leqq y \cup e$ therefore $p^m q^n x^{-1} = p^s q^t$ for some integers s, t which follows $y=p^{m-s}q^{n-t}$.

Here we have $p^m \# q^n$ for every $m, n > 0$. In fact, if $p^m \geqq q^n$ then $p^m \geqq q$ since $q^n \geqq q$, accordingly $p^m \geqq p \cup q$, and from $p < p \cup q \leqq pq$ we get $p \cup q = pq$.

Hence it follows $p^{m-1} \geqq q$. Here if $m-1 > 1$, by repeating this method, we have $p \geqq q$, which is contrary, and $p^m \# q^n$.

Therefore, it is easily seen that $p^m q^n = e$ if and only if $m=0$ and $n=0$ and that $p^a q^b \leqq p^c q^d$ if and only if $a \geqq c$ and $b \geqq d$.

Conversely, on given free abelian group L with two generators p, q if one defines $p^a q^b \geqq p^c q^d$ to mean $a \geqq c$ and $b \geqq d$, then L is a partly ordered set, and a lattice, since $p^m q^n \curlyvee p^a q^b = p^{m \curlyvee a} q^{n \curlyvee b}$, where a, b, m and n are integers.

Moreover, it is easily seen that $x \geqq y$ implies $xz \geqq zy$ for all z.

And given a free abelian group L is a lattice-ordered group, which satisfies the hypotheses of Theorem because $p \mathrel{\overset{*}{\lessgtr}} e, q \mathrel{\overset{*}{\lessgtr}} e$.

We can easily extend above arguments general case by the method as was done in Theorem 2.

THEOREM 3. *If for the identity e of a lattice-ordered group L there exist the elements* x_i $(i=1, 2, \cdots\cdots, n)$ *which* $x_i \mathrel{\overset{*}{\lessgtr}} e$ *for all i and* $y \geqq x_i$ *for some i and any y such that* $y > e$ *in L, then L is a free abelian group with n generarors* x_i $(i=1, 2, \cdots \cdots, n)$, *and in L* $x_1^{a_1} x_2^{a_2} \cdots\cdots x_a^{a_*} \geqq x_1^{b_1} x_2^{b_2} \cdots\cdots x_n^{b_*}$, *if and only if* $a_i \geqq b_i (i=1, 2, \cdots\cdots, n)$. *And vice versa.*

2. Interval topology

Let us, first of all, introduce topological space to given lattice-ordered group L. We denote by $(a]$, $[a)$ and (a,b) the open intervals $\{x\in L|x<a\}$, $\{x\in L|x>a\}$ and $\{x\in L|a<x<b\}$ respectively.

DEFINITION 1. *By the interval topology of a partly ordered set L (in fact, L is a lattice-ordered group), we mean that define by taking the open intervals $(a]$, $[a)$ and (a,b) as a subbase of open sets.*

That is, the family of open intervals

$$S=\{\bigwedge_i \bigvee_\alpha P_{i_\alpha} | P_{i_\alpha} \in \Phi\}, \text{ where } \Phi=\{(a], [a), (a,b) | a, b \in L\},$$

satisfies (1) $\bigvee_\alpha O_\alpha \in S$ for $O_\alpha \in S$,

(2) $\bigwedge_i O_i \in S$ for $O_i \in S$,

(3) $L \in S$,

(4) Empty set $\phi \in S$,

where the suffix i runs on a finite set, and the suffix α on an arbitrary set. And let M be subset of L. By $\overline{M}$ which $\overline{M}=(\bigvee_\alpha M_\alpha)'$ where $M_\alpha \in S$ and $M_\alpha \in M'$ (=the complement of M) we call closure of M.

We also see that if one defines open set O of L to mean $O'=\overline{O}'$ then the family of all open sets of L coincide with S.

And $\quad S=\{\bigwedge_i \bigvee_\alpha (x_{i_\alpha}, y_{i_\alpha}) | x_{i_\alpha}, y_i \in L\}$

In fact, let y be an arbitrary element of $[x)$, there exists no greatest element, we can find at least one element y_α or y_β in L such that $y<y_\alpha$ or $y \# y_\beta$. From $y<y\cup y_\beta$ we have $[x)\subset \bigvee_{x<y}(x, y_\alpha)$, and $x)=\bigvee_{x<y}(x, y_\alpha)$.

Similarly $(x]=\bigwedge_{x<y}(y_\alpha, x)$.

Let E be a subset of L. A point a is called an accumulating point of E, if $a\in \overline{E-a}$, we call a an isolated point of E, if it is not.

On the other hand, we can easily prove the followings.

LEMMA 2. *If $A=\bigwedge_\alpha(x_\alpha, y_\alpha)$, $B=\bigvee_\alpha(x_\alpha, y_\alpha)$, then $Aa=\bigwedge_\alpha(x_\alpha a, y_\alpha a)$, $Ba=\bigvee_\alpha(x_\alpha a, y_\alpha a)$ for every $a\in L$ where $Aa=\{xa|x\in A\}$.*

LEMMA 3. *Let A and B be two subsets of L, then*

$$(A-B)a=Aa-Ba, \quad a(A-B)=aA-aB \text{ for every } a\in L,$$

COROLLARY. *Let A be a subset of L, then $(Aa)'=A'a$ for every $a\in L$.*

And we have also

LEMMA 4. *Let A be a subset of L, then $Aa=\bar{A}a$ for every $a \in L$.*

PROOF. $\bar{A}=L-\bigvee_{\alpha} B_{\alpha}$, for all $B_{\alpha} \in S$ and $B_{\alpha} \subset A'$.

By Lemma 2, 3, $\bar{A}a=L-\bigvee_{\alpha}(B_{\alpha}a)$.

While $\overline{Aa}=L-\bigvee_{\beta} C_{\beta}$ for all $C_{\beta} \in S$ and $C_{\beta} \subset (Aa)'$, and it is sufficient to show that $\bigvee_{\alpha}(B_{\alpha}a)=\bigvee_{\alpha} C_{\beta}$.

Since $B_{\alpha} \subset A'$, we see that $B_{\alpha}a \subset A'a=(Aa)'$ by Corollary 1.

And since $B\alpha \in S$, $B_{\alpha}=\bigwedge_{i}\bigvee_{\gamma}(x_{iy}, y_{iy})$ for some x_{iy} and $y_{iy} \in L$, and hence $B_{\alpha}a=\bigwedge_{i}\bigvee_{\gamma}(x_{iy}a, y_{iy}a)$ by Lemma 2, 3.

Consequently, $B_{\alpha}a \in S$. Therefore, we obtain $B_{\alpha}a \subset \bigvee_{\beta} C_{\beta}$ for any α, i.e., $\bigvee_{\alpha}(B_{\alpha}a) \subset \bigvee C_{\beta}$.

Conversely, $C_{\beta}a^{-1} \subset A'$ because $C_{\beta} \subset A'a$.

Here, by the same way above, we have $C_{\beta}a^{-1} \in S$.

Hence, we obtain $C_{\beta}a^{-1} \subset \bigvee_{\alpha} B_{\alpha}$ namely $C_{\beta} \subset \bigvee_{\alpha}(B_{\alpha}a)$ for any β, i.e., $\bigvee_{\beta} C_{\beta} \subset \bigvee_{\alpha}(B_{\alpha}a)$, which completes the proof.

Now, we prove the following.

THEOREM 4. *L is a lattice-ordered group in which interval topology is intro duced. If there is at least one isolated point of L, then L is descrete.*

PROOF. Let a be an isolated point of L, then $a \in (\overline{L-a})'$. It follows, by Lemma 4, 3,

$e \in (\overline{L-a})'\, a^{-1}=((\overline{L-a})a^{-1})'=(\overline{(L-a)a^{-1}})'=(\overline{L-e})'$

hence the identity e is isolated point of L.

Similarly, $x \in (\overline{L-x})'$ for every element x of L.

COROLLARY 2. *A lattice-ordered group in which interval topology is introduced, is a descrete topological space or the topological space which is composed only of the accumulating points of L.*

3. Topological lattice-ordered group

DEFINITION 2. *A set L of elements is called a topological lattice-ordered group, if*

(1) *L is a lattice-ordered group.*

(2) *L is a topological space.*

(3) *The lattice operations and group operations in L are continuous in the topo-*

logical space.

And by a neighborhood of an element $p \in L$ is meant the subset including an open set $\ni p$.

Then, we have the following.

THEOREM 5 *If there is no element x in a lattice-ordered group L such that $x \gtrless e$, then L is a topological lattice-ordered group under it's interval topology.*

PROOF. It is sufficient to show that the condition (3) of Definition 2 is satisfies in L.

For a neighborhood $U(ab)$ of the element ab of L there exists an open set $O(ab) \ni ab$ such that $O(ab) \subset U(ab)$, and

$$O(ab) = \bigwedge_i \bigvee_\alpha (x_{i\alpha}, y_{i\alpha}),$$

therefore $ab \in (x_{i\alpha}, y_{i\alpha})$ for some α and every i,

$$\text{i.e., } x_{i\alpha} < ab < y_{i\alpha}.$$

On the other hand, there exist the elements $c_{i\alpha}$ and $d_{i\alpha}$ such that $x_{i\alpha} < c_{i\alpha} < ab < d_{i\alpha} < y_{i\alpha}$.

For, if $x_{i\alpha} \gtrless ab$ then $(ab)x_{i\alpha}^{-1} \gtrless e$ which is contrary to the hypotuese of Theorem.

If we consider the open intervals $U(a) = (x_{i\alpha}(a^{-1}c_{i\alpha})^{-1}, y_{i\alpha}(a^{-1}d_{i\alpha})^{-1})$ and $U(b) = (a^{-1}c_{i\alpha}, a^{-1}d_{i\alpha})$, then $U(a)$, $U(b)$ can be neighborhoods of the elements a and b respectively.

In fact, since $x_{i\alpha} < c_{i\alpha}$ and $d_{i\alpha} < y_{i\alpha}$, $x_{i\alpha}(a^{-1}c_{i\alpha})^{-1} > a$, and $a < y_{i\alpha}(a^{-1}d_{i\alpha})^{-1}$ hence $a \in U(a)$ and from $c_{i\alpha} < ab < d_{i\alpha}$, $b \in (a^{-1}c_{i\alpha}, d^{-1}d_{i\alpha})$. Moreover, let x and y be arbitrary elements of $U(a)$ and $U(b)$ respectively, then $xy \in (x_{i\alpha}, y_{i\alpha})$ by transitive law, and it follows $U(a)U(b) \subset U(ab)$, where $U(a) \cdot U(b) = \{xy \mid x \in U(a), y \in U(b)\}$. Now, for a neighborhood $U(x^{-1})$ of the element x^{-1} of L, we have open set $O(x^{-1}) = \bigwedge_i \bigvee_\alpha (x_{i\alpha}, y_{i\alpha}) \ni x^{-1}$ such that $O(x^{-1}) \subset U(x^{-1})$. And $x^{-1} \in (x_{i\alpha}, y_{i\alpha})$ for some α and any i, therefore, $x \in (y_{i\alpha}^{-1}, x_{i\alpha}^{-1}) = U(x)$ which is also a neighborhood of the element x, finally $U^{-1}(x) \subset U(x^{-1})$, where $U^{-1}(x) = \{y^{-1} \mid y \in U(x)\}$. Hence, the group operations in L are continuous in the topological space L.

Finally, in order to show that the lattice operations in L are continuous, let $O(a \cup b) = \bigwedge_i \bigvee_\alpha (x_{i\alpha}, y_{i\alpha}) \ni a \cup b$ be a open subset of a neighborhood $U(a \cup b)$ of the element $a \cup b$ of L. And $a \cup b \in (x_{i\alpha}, y_{i\alpha})$ for some α and any i.

By the above method, we obtain that for the elements $x_{i\alpha}$, $y_{i\alpha}$, there exist the elements $c_{i\alpha}$ and $d_{i\alpha}$ of L such that

$$x_{i\alpha} < c_{i\alpha} < a \cup b < d_{i\alpha} < y_{i\alpha}.$$

And if we consider the open intervals $U(a) = (c_{i\alpha} b^{-1}(a \cap b),\ d_{i\alpha})$, and $U(b) = (c_{i\alpha} a^{-1}(a \cup b),\ d_{i\alpha})$, then we also see tnat $U(a)$, $U(b)$ can be neighborhoods of the elements a, b respectively. For, since $c_{i\alpha} < a \cup b$ implies $a^{-1} c_{i\alpha} b^{-1} < b^{-1} \cup a^{-1} = (a \cap b)^{-1}$, $c_{i\alpha} b^{-1}(a \cap b) < a < d_{i\alpha}$, and $c_{i\alpha} a^{-1}(a \cap b) < b < d_{i\alpha}$.
Moreover, $U(a) \cup U(b \subset U(a \cup b)$, in fact, let x, y be the elements of $U(a)$ and $U(b)$ respectively, then

$$c_{i\alpha} b^{-1}(a \cap b) \cup c_{i\alpha} a^{-1}(a \cap b) \leqq x \cup y \leqq d_{i\alpha},$$
$$c_{i\alpha}(b^{-1} \cup a^{-1})(a \cap b) \leqq x \cup y \leqq d_{i\alpha},$$
$$\text{i.e.,}\quad c_{i\alpha} \leqq x \cup y \leqq d_{ix},$$

and hence $x \cup y \in \bigwedge_i \bigvee_\alpha (x_{i\alpha},\ y_{i\alpha}) \subset U(a \cup b)$.
Hence the lattice operations in L are continuous in the topological space L.

August 21, 1957,
Graduate School,
Kyungpook University

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