NOTE ON A LATTICE-ISOMORPHISM BETWEEN FINITE GROUPS WITH COMPLETE PARTITIONS

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1. Introduction

By a *partition* of a group G we mean a system $\{H_i\}$ of subgroups of G in which each element of G except the identity is contained in one and only one of the subgroups H_i, which are called components of the partition. A partition of G is called *complete* when all of its components are cyclic. A group with a complete partition is called *completely decomposable*, briefly, c. d.

A set $L(G)$ of all subgroups of a group G is said to be a lattice provided there exist the join $A \cup B$ of the elements A and B of $L(G)$ which is a subgroup generated by A and B, and the meet $A \cap B$ a set intersection of A and B. Here by the notation $\bigvee$ we mean a set union in order to distinguish from the join $\cup$. By a lattice-isomorphism ϕ of a group G onto a group G' we mean a one-to-one correspondence between $L(G)$ and $L(G')$ which preserves all joins and meets.

In this paper we shall deal with finite groups with complete partition which have been considered by P. Kontorovitch [3](*) and M. Suzuki [4]. We are going to study the structure of a lattice-isomorphic image of a group with a complete partition in 2 and of that of a nilpotent group with a complete partition in 3, and in 4 we give an example of a non-solvable, non-simple group with a complete partition. In this paper we assume that G' always a lattice-isomorphic image of a group G by the ϕ without noticing it in each article.

2. Groups with complete partitions

We now investigate a lattice-isomorphic image of a group with a complete partition.

LEMMA 1. *The lattice-isomorphic image of any cyclic subgroup of a group is also cyclic.*

PROOF. Let A be any cyclic subgroup of a group and A' the lattice isomorphic image of A. The image H^{ϕ} of an arbitrary subgroup H of A is in $L(G')$ and is a subgroup of $A'=A^{\phi}$. And $L(A)$ is lattice-isomorphic with $L(A')$ by the ϕ.

(*) Numbers in parentheses refer to the references at the end of the paper.

Since the lattice of all subgroups is distributive if and only if the group is cyclic [1], $L(A)$ is a distributive lattice, therefore A' is cyclic.

LEMMA 2. *A lattice-isomorphic image of a group with a partition has also the partition consisting of the same number of components.*

PROOF. Let $G=\vee_{i=1}^{n} H_i$ be a partition of G. Since every element of $L(G)$ has a image which is a subgroup of G' the H_i' are subgroups of G', and hence $\vee_{i=1}^{n} H_i^{\phi}$ is included in G'. A cyclic subgroup $\{a'_i\}$ generated by an arbitrary element a' of G' has a preimage $\{a\}$ such that $\{a_i^{\phi}\}=\{a'i\}$ by lemma 1, here $a \in H_i$ for some i and $\{a\}$ is a subgroup of H_i, therefore $a' \in H_i^{\phi}$. Hence $\vee_{i=1}^{n} H_i^{\phi}$ coincides with G'. While we know readily $H_i^{\phi} \cap H_j^{\phi}=e'$ for all distinct i, j, so $G' =\vee_{i=1}^{n} H_i^{\phi}$ is a partition of G'.

And we have the following result from lemma 1 and lemma 2:

THEOREM 1. *A lattice-isomorphic image of a completely decomposable group is completely decomposable.*

As a lattice--isomorphism preserves the solvable property of a group,

COROLLARY 1. *The lattice-isomorphic image of a solvable c.d. group is also a solvable c.d. group.*

Here, as to a lattice-homomorphism, we merely point out that the lattice-homomorphic image G' of a c.d. group G is c.d. too. In fact, according to Whitman [2], for any cyclic subgroup $\{a'\}$ of G' there exists a cyclic subgroup $\{a\}$ of G such that $\{a\}$ is mapped lattice-homomorphically upon $\{a'\}$. And G' is c.d. by the same way in theorem 1.

3. Nilpotent groups with complete partitions

LEMMA 3. *Any subgroup of c.d. group is c.d.*

PROOF. Let $G=\vee_i H_i$ be a complete partition of G. For any subgroup A of G, $A=A\cap \vee_i H_i=\vee_i A\cap H_i$. As $A\cap H_i$ is a subgroup of a cyclic group H_i, $A\cap H_i$ is a cyclic subgroup for all i. By the definition of components H_i, $A=\vee_i A\cap H_i$ be a complete partition of A.

LEMMA 4. *A nilpotent c.d. group is either cyclic or of prime power order* [3].

PROOF. Let $G=\vee_i H_i$ be a complete partition of a nilpotent c.d. group. Suppose G is not of prime power order, we shall prove that G is cyclic. Let a be an element of G of prime order p^n. We may think a is contained in one component H_1. For a prime factor q of order of G which is different from p, we take an arbitrary element b of order q^n. Since G is nilpotent, $ab=ba$ and the subgroup $\{a, b\}$ generated by a and b coincides with a cyclic subgroup $\{ab\}$. So $\{a, b\}$ is contained in H_1, hence H_1 contains the q-Sylow group of G by lemma 3. Since a nilpotent group G is a direct product of Sylow groups, we obtain finally $G=H_1$ and G is cyclic.

THEOREM 2. *Let G be a nilpotent c.d. group, not $(p \cdots\cdots p)$-type Abelian group, then the image G' is a nilpotent c.d. group.*

PROOF. If G is cyclic, our theorem is evidently true by lemma 1. And suppose G is a nilpotent proper c.d. group (that is, consisting of more than one component), not $(p \cdots\cdots p)$-type Abelian group. Then G is a p-group. Since such a p-group has a lattice-isomorphic image which is a p-group [5], G' is a p-group and hence a nilpotent c.d. group by theorem 1.

Now we shall consider the exceptional case in theorem 2.

LEMMA 5. *The lattice-isomorphic image of any subgroup of prime order p in G has a prime order q.*

PROOF. Let A be a subgroup of a prime order p in G, then the image A' of A has no proper subgroup. On the other hand, let the order of A' be $q_r \cdot l$, where q is a prime number, A' has a subgroup of order q^s for any s $(0<s\leqq r)$ because A' is cyclic. And both r and l must be 1.

LEMMA 6. *If G is a $(p \cdots\cdots p)$-type Abelian group, then the image G' is a group in which every element has a prime order.*

PROOF. Since G is a group whose element is always of order p, G is c.d. And G' has a complete partition whose component has prime order by lemma 5. Hence each element of G' must have a prime order.

A group is nilpotent if and only if any two elements whose orders are relatively prime are commutative, and we can conclude.

THEOREM 3. *The lattice-isomorphic image of a $(p \cdots\cdots p)$-type Abelian group is nilpotent when and only when arbitrary two elements whose orders are distinct are commutative.*

Now we determine the structure of a lattice-isomorphic image of a group which has only elements of order p and is not a $(p \cdots\cdots p)$-type Abelian group:

THEOREM 4. *A group G of order $>p$ whose element has always order p, not*

$(p\cdots\cdots p)$-type Abelian group, is isomorphic with its lattice-isomorphic image G'.

PROOF. G is a c.d. p-group, not cyclic and all components H_i have the same order p. And G' is a p-group of order p^λ. By lemma 5 orders of the H_i^ϕ are prime numbers and hence the p, therefore every element of G' is also of order p.

In the preceding theorem, particularly G, not commutative, is a Hamiltonian group. For any subgroup of G is a set-union of some components of G which are normal subgroups of a p-group G. Hence G is of order 2^α as well as G' [6] and the ϕ is a normal lattice-isomorphism——the image of a normal subgroup is a normal subgroup.

4. An example

We here give an example of a non-solvable, non-simple c.d. group, say a symmetric group S_n of n-digits. In fact let $\{\sigma\}$, $\{\tau\}$ be any two maximal cyclic subgroups of S_n. If they have $\sigma^\alpha=\tau^\beta\neq 1$ in common, i.e., $(i_1\cdots\cdots i_s)^\alpha\cdots\cdots(j_1\cdots\cdots j_t)^\alpha=(k_1\cdots\cdots k_p)^\beta\cdots\cdots(l_1\cdots\cdots l_q)^\beta$ as products of powers of cycles, then there are at least two distinct cycles such that $(i_1\cdots\cdots i_s)^\alpha=(k_1\cdots\cdots k_p)^\beta\neq 1$. And $\{\sigma\}\cap\{\tau\}=1$, so S_n is c.d.

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REFERENCES

[1] O. Ore: *Structures and group theory II*, Duke Jour. 4 (1936), 247-269.

[2] P. Whitman: *Groups with cyclic groups as a lattice-homomorph*. Ann. of Math., 49 (1948).

[3] P. Kontorovitch: *Sur la representation d'un groupe fini sous la formed'une smme directede sous-groupes I, II*. Rec. Math. (Nat. Sbornik) 5, 7 (47, 49).

[4] N. Suzuki: *On the finite group with a complete partition*, Jap. Jour. Math. (1950) 165-185.

[5] M. Suzuki: *On the lattice of subgroups of finite gioup*, Sugaku, (1950) 189-200.

[6] Dedekind: *Math. Ann.* 48. (1897)